THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and seventy-ninth regular meeting of the Society was held in New York City on Saturday, October 30, 1915. The attendance at the morning and afternoon sessions included the following fifty members:

Professor R. C. Archibald, Dr. A. A. Bennett, Mr. D. R. Belcher, Professor E. G. Bill, Professor W. J. Berry, Professor G. D. Birkhoff, Professor E. W. Brown, Dr. T. H. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Dr. H. B. Curtis, Mrs. E. B. Davis, Dr. C. R. Dines, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor W. B. Fite, Professor O. E. Glenn, Dr. G. M. Green, Professor C. N. Haskins, Professor H. E. Hawkes, Professor L. A. Howland, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Dr. H. F. MacNeish, Dr. L. C. Mathewson; Professor F. M. Morgan; Mr. George Paaswell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Mr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor W. B. Stone, Mr. H. S. Vandiver, Professor Oswald Veblen, Dr. Mary E. Wells, Professor H. S. White, Miss E. C. Williams, Professor A. H. Wilson, Professor J. W. Young.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Vice-President Oswald Veblen. The Council announced the election of the following persons to membership in the Society: Mr. D. R. Belcher, Columbia University; Professor J. W. Calhoun, University of Texas; Professor Sarah E. Cronin, State University of Iowa; Mr. C. A. Epperson, State Normal School, Kirksville, Mo.; Dr. Olive C. Hazlett, Radcliffe College; Mr. C. M. Hebbert, University of Illinois; Miss Goldie P. Horton, University of Texas; Professor W. S. Lake, Bendigo School of Mines and Industries, Australia; Mr. D. H. Leavens, College of Yale in China; Mr. C. T. Levy, University of California; Dr. F. W. Reed, University of Illinois; Professor L. H. Rice, Syracuse University; Mr. J. F. Ritt, Columbia University; Professor

D. M. Y. Sommerville, Victoria University College, Wellington, N. Z.; Miss L. R. Stoughton, Rosemary Hall School, Greenwich, Conn.; Dr. C. E. Wilder, Pennsylvania State College; Mr. A. R. Williams, University of California; Dr. L. T. Wilson, University of Illinois; Dr. F. E. Wright, U. S. Geological Survey. Four applications for membership in the Society were received.

The Council submitted a list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual election. A committee was appointed to audit the accounts of the Treasurer for the current year.

The dinner in the evening, always a pleasant feature of the meetings, was attended by twenty-one members and friends.

The following papers were read at this meeting:

(1) Dr. G. A. PFEIFFER: "Existence of divergent solutions of the functional equations $\varphi[g(x)] = a\varphi(x)$, $f[f(x)] = g(x)$, where $g(x)$ is a given analytic function, in the irrational case."

(2) Professor C. N. HASKINS: "On the extremes of bounded summable functions and the distribution of their functional values."

(3) Dr. G. M. GREEN: "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface. Second memoir."

(4) Dr. G. M. GREEN: "The linear dependence of functions of several variables."

(5) Mr. A. R. SCHWEITZER: "On the dependence of algebraic equations upon quasi-transitiveness."

(6) Professor H. S. CARSLAW: "A trigonometrical sum and the Gibbs phenomenon in Fourier's series."

(7) Professor W. F. OSGOOD: "On a sufficient condition for a non-essential singularity of a function of several complex variables."

(8) Dr. DUNHAM JACKSON: "Singular points of functions of several complex variables."

(9) Professor W. F. OSGOOD: "On functions of several complex variables."

(10) Professor L. P. EISENHART: "Envelopes of rolling and transformations of Ribaucour."

(11) Professor W. B. FITE: "Note on linear homogeneous differential equations of the second order."

(12) Mr. H. S. VANDIVER: "Note on the distribution of quadratic residues."

(13) Professor G. D. BIRKHOFF: "A theorem concerning the singular points of ordinary linear differential equations."

(14) Professor H. S. WHITE: "Closed systems of sevens in a 3-3 correspondence."

(15) Professor W. R. LONGLEY: "Note on a theorem on envelopes."

(16) Mr. A. R. SCHWEITZER: "On the dependence of algebraic equations upon quasi-transitiveness. Second paper."

(17) Mr. A. R. SCHWEITZER: "A new functional characterization of the arithmetic mean."

Professor Carslaw's paper was communicated to the Society through Professor Bôcher. In the absence of the authors Professor Osgood's second paper was read by Professor Birkhoff and the papers of Mr. Schweitzer, Professor Carslaw, Professor Longley, and the first paper of Professor Osgood were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The first functional equation considered in Dr. Pfeiffer's paper arises in the consideration of the conformal equivalence of curvilinear angles in the so called irrational case. In this case a transformation can always be found formally. The author shows that the series giving the transformation may be divergent. The second functional equation here considered determines the bisectors of an irrational curvilinear angle as defined by Kasner, and it is proved that such an angle may have two, one, or no bisectors.

Stated specifically the theorems obtained are

1. There exists an analytic function $g(x) \equiv a_1x + a_2x^2 + \cdots$ ($|a_1| = 1$, $a_1^n \neq 1$, $n = 1, 2, \cdots$) such that every formal solution $\phi(x) = c_1x + c_2x^2 + \cdots$ ($c_1 \neq 0$) of the functional equation $\phi[g(x)] = a_1\phi(x)$ is divergent for all values of $x \neq 0$.

2. Given the functional equation $f[f(x)] = g(x)$. Then $g(x) \equiv a_1x + a_2x^2 + \cdots$ ($|a_1| = 1$, $a_1^n \neq 1$, $n = 1, 2, \cdots$) may be taken such that the given equation has (*a*) no solution analytic about the origin, (*b*) only one such solution, (*c*) two such solutions. The number of solutions cannot be greater than two.

The author is indebted to Professor Birkhoff for suggestions which led to the above development.

2. Professor Haskins's paper presents, first, a method of the integral calculus, as contrasted with the classic methods of the differential calculus, for the determination of the extrema extremorum of bounded Lebesgue-integrable functions; and second, certain results concerning what may be termed the statistical distribution of functional values.

The methods used are allied to those of Laplace-Darboux-Stieltjes for the determination of "functions of large numbers." Some of the results (for the case of continuous functions and Riemann integrals) have already been presented to the Society. The full significance of the results appears however only when the Lebesgue integral is used, for the reason that these results relate to certain constants and certain analytic functions which serve to divide all Lebesgue-integrable functions into classes such that all the functions of each class and only those have the same defining elements of their Lebesgue integrals.

3. In a recent paper,* Dr. Green laid the foundations for a purely projective theory of conjugate nets on a curved surface, his present paper forming a continuation of this study. The first part of the memoir contains a canonical development of the non-homogeneous coordinates of the surface in the neighborhood of a point, which was the subject of a previous communication to the Society.† The discussion is completed in the present paper. The tetrahedron of reference which gave rise to the canonical development has as two of its edges lines which Wilczynski‡ has called the axis and the ray of a point of the surface. The second part of Dr. Green's paper contains an investigation of the properties of the conjugate net expressed in terms of the axis and ray congruences. A new net of curves is introduced, viz., the associate conjugate net, the tangents to the two curves of which at any point separate harmonically the tangents to the curves of the original conjugate net. Any conjugate net has one and only one associate conjugate net. It appears that the theory of a conjugate net is very conveniently investigated through the consideration thereof in connection with its associate conjugate net. A discussion from this point of view leads to many interesting theorems, some of

* *Amer. Jour. of Mathematics*, vol. 37 (1915), pp. 215–246.
† See this BULLETIN, vol. 20, no. 8 (May, 1914), p. 397.
‡ *Transactions*, vol. 16 (1915), pp. 311–327.

them already known; in fact, the study of a conjugate net, its associate conjugate net, and the axis and ray congruences for both, seems to afford a powerful method for describing geometrically projective properties of the conjugate net. An application is made at the end of the paper to the conjugate nets which Bianchi has called isothermally conjugate. For such a net, for instance, the associate conjugate net is also isothermally conjugate.

4. For a set of functions of a single real variable, it is well known that in certain cases the vanishing of the wronskian is a sufficient condition for linear dependence. The wronskian arises naturally out of the study of a single ordinary homogeneous linear differential equation, the characteristic property of which is that any solution is expressible linearly, with constant coefficients, in terms of a fundamental set of solutions. The natural generalization of this kind of ordinary differential equation is the completely integrable system of homogeneous linear partial differential equations, the solutions of which have the same property. A year ago, Dr. Green* gave a generalization for such systems of the notion of wronskian, but this can be done only when the completely integrable system is given. It is the object of Dr. Green's second paper to supply a sufficient condition for the linear dependence of a set of functions of several independent real variables. There is no determinant of fixed form analogous to the wronskian, but the criteria given are very general, including by specialization those concerning the wronskian for functions of a single variable. Application of the theorems is made to completely integrable systems, yielding results analogous to those for ordinary homogeneous linear differential equations.

5. In the first part of Mr. Schweitzer's paper it is shown that if

$$(1)\quad f(x_1 + y_1, x_2 + y_2, \cdots, x_{n+1} + y_{n+1}) = f(x_1, x_2, \cdots, x_{n+1}) + f(y_1, y_2, \cdots, y_{n+1}),$$

$$(2)\quad f\{f(t_1, t_2, \cdots, t_n, x_1)f(t_1, t_2, \cdots, t_n, x_2) \cdots f(t_1, t_2, \cdots, t_n, x_{n+1})\} = f\left(\frac{x_2}{c_1}, \frac{x_3}{c_2}, \cdots, \frac{x_{n+1}}{c_n}, \frac{x_1}{c_{n+1}}\right),$$

* See this BULLETIN, vol. 21, no. 4 (Jan., 1915), p. 162.

then

$$f(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n} [c'_{n-i+1}\xi^{n-i+2} \cdot x_i] + \xi x_{n+1},$$

where

$$c'_{n-i+1} = c_i \cdot c_{i+1} \cdots c_n, \qquad \frac{1}{c_{n+1}} = c_1 \cdot c_2 \cdots c_n \cdot \xi^{n+1},$$

$$1 + \sum_{i=1}^{n} c'_{n-i+1} \cdot \xi^{n-i+1} = 0.$$

In the second part of the paper an alternative method of inducing algebraic equations of the nth degree is discussed. The latter treatment is based on the equation (1) in conjunction with functional equations derived from the relation

$$f\{f_1(t_1^{(1)}, t_2^{(1)}, \cdots, t_n^{(1)}, x_1), f_2(t_1^{(2)}, t_2^{(2)}, \cdots, t_n^{(2)}, x_2) \cdots f_{n+1}(t_1^{(n+1)}, t_2^{(n+1)}, \cdots, t_n^{(n+1)}, x_{n+1})\} = \psi(x_1, x_2, \cdots, x_{n+1})$$

(or analogous relations obtained by the homologous transposition of the x's on the left side) by substituting suitable x's for some or all of the t's and assuming the remaining t's (if any) with the same subscript to be identical. In the simplest cases the following theorems are valid:

I. $f\{f(x_3, x_1, t), f(x_1, x_2, t), f(x_2, x_3, t)\} = f(x_2, x_1, x_3)$ implies

$$f(x_1, x_2, x_3) = \frac{1}{2}x_1 + \frac{1 \pm \sqrt{13}}{4}x_2 - \frac{3 \pm \sqrt{13}}{4}x_3.$$

II. $f\{f(x_2, x_3, x_1), f(x_3, x_1, x_2), f(x_1, x_2, x_3)\} = f(x_2, x_3, x_1)$ implies

$$f(x_1, x_2, x_3) = \tfrac{1}{3}(2x_1 - cx_2 - c^2x_3) \quad \text{or} \quad \tfrac{1}{3}(x_1 + cx_2 + c^2x_3) \quad \text{or} \quad \tfrac{1}{3}(x_1 + x_2 + x_3),$$

where $1 + c + c^2 = 0$.

III. $f\{f(x_3, t, x_1)f(x_1, t, x_2)f(x_2, t, x_3)\} = f(x_2, x_1, x_3)$ implies $f(x_1, x_2, x_3) = -c^2x_1 - c^4x_2 + cx_3$, where $c^3 + c - 1 = 0$.

IV. $f\{f(t, x_3, x_1)f(t, x_1, x_2)f(t, x_2, x_3)\} = f(x_2, x_1, x_3)$ implies

$$f(x_1, x_2, x_3) = -\frac{1}{2}x_1 + \frac{1 \pm \sqrt{5}}{4}x_2 + \frac{1 \mp \sqrt{5}}{4}x_3$$

or $2x_1 - x_2 - x_3$.

In deriving the preceding results the operation of differentiation is not employed.

6. In Professor Carslaw's paper the Gibbs phenomenon in Fourier series is deduced from the behavior of the approximation curves for the Fourier series

$$2(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots). \tag{1}$$

In the interval $-\pi < x < \pi$, this series represents the function defined by the following equations:

$$f(-\pi) = f(\pi) = f(0) = 0, \qquad f(x) = -\tfrac{1}{2}\pi(-\pi < x < 0),$$
$$f(x) = \tfrac{1}{2}\pi(0 < x < \pi).$$

Bôcher, in his discussion of the Gibbs phenomenon (*Annals of Mathematics*, (2), volume 7, 1906, and *Crelle's Journal*, volume 144, 1914), uses the series

$$\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots. \tag{2}$$

He deals with the maxima and minima of $R_n(x)$, the remainder after n terms.

Gronwall (*Mathematische Annalen*, volume 72, 1912) discusses the question of the behavior of the maxima and minima of $S_n(x)$, the sum of n terms of the series (2), and deduces the Gibbs phenomenon from the properties of $S_n(x)$. But both his results and the method by which they are obtained are somewhat complicated.

In the present paper the behavior of the maxima and minima of $S_n(x)$ for the series (1) is examined. The properties of the approximation curves $y = S_n(x)$ are much simpler for (1) than for (2). From the results obtained, the Gibbs phenomenon for the series (1) follows immediately. The extension to the general case is simple.

7. Professor Osgood's first paper contains the following theorem. In order that a function $F(z_1, \cdots, z_n)$ have a non-essential singularity in a point $(z) = (a)$ it is sufficient:

(1) that $F(z_1, \cdots, z_n)$ be uniquely defined and analytic at all points of a certain neighborhood Σ of the point (a), which do not lie on a locus defined by an equation of the form $H(z_1, \cdots, z_n) = 0$, where $H(z_1, \cdots, z_n)$ is analytic in the point (a) and vanishes there, but does not vanish identically.

(2) that $F(z_1, \cdots, z_n)$ in general become infinite in those points of the above locus which lie in Σ, the excepted points being those which lie simultaneously on a second locus, $G(z_1, \cdots, z_n) = 0$, where $G(z_1, \cdots, z_n)$ is subject to the same conditions as $H(z_1, \cdots, z_n)$, and where furthermore G and H have no common factor in the point $(z) = (a)$.

The proof is given in the author's Funktionentheorie, volume II, chapter 3, § 2.

8. It is a familiar property of functions of a single complex variable that if $f(z)$ has a pole at the point a, it can be written throughout the neighborhood of this point in the form $f(z) = \varphi(z)/(z - a)^m$, where m is a suitable positive integer and $\varphi(z)$ is a function of z which is analytic at a and does not vanish there. Dr. Jackson gives one of the possible generalizations of this theorem to functions of several variables, differing somewhat in its hypotheses from one which has been obtained by Professor Osgood. It is assumed that $f(z_1, z_2, \cdots, z_n)$ is analytic throughout the neighborhood of $(a_1, a_2, \cdots, a_n)$, except for singularities which satisfy a certain requirement of continuity in their distribution, and which are of such a nature that f is analytic except for poles when regarded as a function of z_1 alone; and it is shown that under these conditions f can be expressed as the quotient of two functions, each analytic in the neighborhood of $(a_1, a_2, \cdots, a_n)$. The denominator is of course no longer merely a power of a linear factor, in general, and the numerator may vanish at points of the neighborhood in question. The proof is closely related in method to well-known proofs of Weierstrass's theorem of factorization.

9. Let $x_i = \varphi_i(u_1, \cdots, u_{n-1})$, $i = 1, \cdots, n$, be a set of functions, each analytic at the origin $(u) = (0)$, and vanishing there. Will the points $(x_1, \cdots, x_n)$ of the locus thus defined lie on a hypersurface $\Omega(x_1, \cdots, x_n) = 0$, where Ω is analytic at the origin and vanishes there, but does not vanish identically? If $n = 2$, it is well known that the answer is affirmative. Professor Osgood shows by an example that, when $n > 2$, the corresponding theorem is false.

Let the functions $f_i(u_1, \cdots, u_n)$, $i = 1, \cdots, n$, be analytic at the point $(u) = (b)$, and let $f_i(b_1, \cdots, b_n) = a_i$. Let T be an arbitrary neighborhood of the point $(a) = (a_1, \cdots, a_n)$.

Finally, let the Jacobian of the functions f_i vanish identically. Then it is well known that there exists a point (a') of T and a function $\Omega(x_1, \cdots, x_n)$ analytic at (a') and not identically $= 0$, and furthermore such that, if x_i be replaced by f_i, Ω then goes over into a function of $(u_1, \cdots, u_n)$ which vanishes identically in these arguments. Can the point (a') be taken at (a)?

Professor Bliss has shown that it can when $n = 2$. It is shown in the present paper that when $n > 2$ this is not in general possible.

Thirdly, the tentative generalization of Weierstrass's theorem of factorization, concerning which Professor Osgood raised a query in the Madison Colloquium Lectures, page 185, is shown to be impossible when $n > 1$.

Finally, a function z of x and y defined by the equation $G(x, y, z) = 0$, where G is analytic at the origin and vanishes there, but does not vanish identically, may have a natural boundary in every neighborhood of the origin.

10. In a note published in volume 23 of the *Rendiconti dei Lincei* Bianchi has defined as an envelope of rolling the surface enveloped by a plane invariably fixed with respect to a surface S_0 as the latter rolls over an applicable surface S, which Bianchi calls the surface of support. He shows that, given any surface Σ, the problem of finding pairs of applicable surfaces S_0 and S such that Σ is an envelope of rolling as S_0 rolls over S reduces to the integration of a partial differential equation of the second order and to quadratures. Two surfaces are said to be in the relation of a transformation of Ribaucour when they constitute the envelope of a two-parameter family of spheres such that the lines of curvature correspond on the two surfaces, corresponding points being on the same sphere. Professor Eisenhart has shown that the necessary and sufficient condition that either surface of two so related be an envelope of rolling with the locus of centers of the spheres for surface of support is that the correspondence of the spherical representations of the lines of curvature of the two surfaces in the relation of a transformation of Ribaucour be conformal. This latter condition is satisfied in case these spherical representations are isothermal, and only in this case. Any surface with isothermal spherical representation of its lines of curvature admits of transforma-

tions of Ribaucour of this kind, as the author has shown in several former papers.

11. Professor Fite determines certain intervals within which a solution of the equation $y'' + py' + qy = 0$ and its derivative cannot both vanish. The argument is based upon the obvious fact that such a solution and its derivative cannot both vanish in any interval within which q is negative.

12. In the BULLETIN for November, 1915, Mr. Vandiver considered some theorems which he applies in the present note to problems regarding the distribution of quadratic residues for a prime modulus. Some special quadratic forms are also considered.

13. Professor Birkhoff proves that if $Y(x)$ is the matrix solution of a linear differential system with singular point of rank $q + 1$ at $x = \infty$, then, if $\varphi(x) - x$ vanishes to the qth order at $x = \infty$, the matrix $A(x)$ defined by the equation $Y(\varphi(x)) = A(x)Y(x)$ is composed of elements analytic at $x = \infty$. The converse is also proved. This note is to appear in the *Proceedings of the National Academy of Sciences.*

14. In a previous note Professor White had proved a property of seven points on a gauche cubic curve: that certain sets of seven planes determined by them osculate other cubic curves. In the present note it is established that such a set of seven points on the one curve and seven osculating planes of the other curve is variable while the curves remain fixed, every point of the first curve belonging to one such set of seven.

15. The theorem in question was published by Risley and MacDonald in the *Annals of Mathematics*, second series, volume 12, page 86, and gives certain criteria for the existence and non-existence of an envelope of the system of curves $y = f(x, \alpha)$ in the neighborhood of a point (x_0, y_0). In the present note Professor Longley points out that the conclusion of the theorem does not follow from the argument and gives some examples which show that it is impossible from the hypotheses of the theorem to draw any conclusion concerning the non-existence of an envelope in the neighborhood of the point in question.

16. In a previous communication Mr. Schweitzer defined a (symmetric) group of functional equations on $n+1$ variables ($n = 1, 2, 3, \cdots$) viz.,

$$f\{f(t_1, t_2, \cdots, t_n, x_1), f(t_1, t_2, \cdots, t_n, x_2), \cdots f(t_1, t_2, \cdots, t_n, x_{n+1})\} = f\{x_{i_1}, x_{i_2}, \cdots, x_{i_{n+1}}\}.$$

In the present paper the group of equations is studied which results from the preceding group by replacing, in the above relation, the x_{i_k} by the functions $\alpha_k(x_{i_k})$ $[k=1, 2, \cdots, (n+1)]$. In particular, the relation corresponding to the substitution $[1, 2, \cdots, (n+1)]$ implies that there exists a function $\psi(x)$ such that

$$\psi\alpha_k(x) = m_k\psi(x) + p_k,$$

$$\psi f(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n} \frac{\xi^{n-i+2} \cdot \psi(x_i)}{m_i \cdot m_{i+1} \cdots m_n} + \xi\psi(x_{n+1}) + l,$$

where ξ, m_k, p_k, l are constants such that

$$\xi^{n+1} = m_1 \cdot m_2 \cdots m_n \cdot m_{n+1}, \quad \sum_{i=1}^{n} \frac{\xi^{n-i+2}}{m_i m_{i+1} \cdots m_n} + \xi = 0,$$

and

$$\sum_{i=1}^{n} \frac{\xi^{n-i+2} p_i}{m_i m_{i+1} \cdots m_n} + \xi p_{n+1} = 0.$$

By making special assumptions concerning the functions $\alpha_k(x)$ the function $\psi(x)$ can be particularized.

17. The postulates constructed by Mr. Schweitzer for the arithmetic mean are as follows:

1. $f\{x_1 + y_1, x_2 + y_2, \cdots, x_{n+1} + y_{n+1}\} = f(x_1, x_2, \cdots, x_{n+1}) + f(y_1, y_2, \cdots, y_{n+1})$.

2. $f(x_1, x_2, \cdots, x_{n+1}) = f(x_2, x_3, \cdots, x_{n+1}, x_1)$.

3. $f\{f(x_1, x_2, \cdots, x_n, x_{n+1}), f(x_2, x_3, \cdots, x_{n+1}, x_1), \cdots f(x_{n+1}, x_1, x_2, \cdots, x_n)\} = f(x_1, x_2, \cdots, x_n, x_{n+1})$.

F. N. COLE,
Secretary.

THE TWENTY-SEVENTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-seventh regular meeting of the San Francisco Section of the Society was held at Stanford University on November 20, 1915. Sixteen persons were present, including the following members of the Society:

Professor R. E. Allardice, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor E. W. Ponzer, Professor T. M. Putnam.

The chairman of the Section, Professor Haskell, presided at the opening of the meeting; the chairman-elect, Professor Allardice, then took the chair. The following officers were elected for the ensuing year: chairman, Professor Allardice; secretary, Dr. Buck; programme committee, Dr. Buck and Professors Blichfeldt and Manning.

It was voted that the time and place of the spring meeting be determined by the executive committee. The usual luncheon was served.

The following papers were presented at this meeting:

(1) Dr. B. A. BERNSTEIN: "A simplification of the Whitehead-Huntington set of postulates for Boolean algebras."

(2) Professor H. F. BLICHFELDT: "Some diophantine approximations."

(3) Dr. FRANK IRWIN: "A convergent series derived from the harmonic series."

(4) Dr. H. N. WRIGHT and Dr. FRANK IRWIN: "Some properties of the curves, $y =$ a polynomial in x."

(5) Mr. T. A. PIERCE: "The numerical factors of the arithmetic forms $\prod_{i=1}^{n} (1 \pm \alpha_i^m)$."

(6) Professor D. N. LEHMER: "Concerning certain binary forms of higher degrees than the second whose prime divisors are of the form $nx \pm 1$."

(7) Professor M. W. HASKELL: "A note on integration."

Dr. Wright was introduced by Dr. Irwin and Mr. Pierce by Professor Lehmer.

Abstracts of the papers follow.

1. The best set of postulates for Boolean logic, considered from the point of view of elegance and pedagogy, is that given by Whitehead and made rigorous by Huntington. The number of propositions in this set is ten and the number of special elements postulated three. Dr. Bernstein, by replacing three postulates of this set by a new proposition, while retaining elegance and naturalness, reduces the number of postulates to eight and the number of postulated special elements to one.

2. Let $X, Y, Z, \cdots$ be n linear homogeneous functions of n variables $x, y, z, \cdots$, having a determinant $\Delta \neq 0$. Minkowski has proved that such integers (not all zero) can be substituted for $x, y, z, \cdots$ that the sum $|X| + |Y| + |Z| + \cdots \leqq k\Delta^{1/n}$, where k is a function of n, independent of the coefficients involved in $X, Y, Z, \cdots$ (Geometrie der Zahlen, Leipzig, 1910). By applying the processes explained in "A new principle in the geometry of numbers," *Transactions*, 1914, Professor Blichfeldt obtains a much lower value for k, except for small values of n, than those given by Minkowski and by the author in the above article in the *Transactions*.

3. In extension of a result of Dr. Kempner in the *American Mathematical Monthly* for February, 1914, Dr. Irwin shows that we shall obtain a convergent series if we omit from the harmonic series all terms whose denominators contain the digits 9, 8, 7, $\cdots$ *each* more than a given number $a, b, c, \cdots$ of times respectively.

4. It is shown by Dr. Wright and Dr. Irwin that if to the curve $y = f(x)$, where $f(x)$ is a polynomial, all the tangents be drawn from a given point, then (1) the sum of their slopes and (2) the sum of the abscissas of their points of contact are independent of the ordinate of the given point. A remarkable line associated with the curve is noticed. Finally a graphical construction is given for the imaginary roots of a cubic equation and of certain biquadratics.

5. In volume 1 of the *American Journal of Mathematics* Lucas published a paper dealing with the properties of numbers given by $\dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $\alpha^n + \beta^n$, where α and β are the roots of

a quadratic equation having integral coefficients. Carmichael has given a different and more exhaustive treatment of these numbers in the *Annals of Mathematics*, 1913. In the present paper Mr. Pierce obtains somewhat similar results for numbers given by the forms $\prod_{i=1}^{n}(1 \pm \alpha_i^m)$, where the α_i denote algebraic integers defined as the roots of an nth degree equation. The forms of the factors of $\prod_{i=1}^{n}(1 - \alpha_i^m)$ are determined by use of algebraic number theory, and this perhaps constitutes the most novel result of the work.

6. Lucas has developed the theory of the prime divisors of the functions $U_n = (a^n - b^n)/(a - b)$ and $V_n = a^n + b^n$, where a and b are the roots of a quadratic equation (*American Journal of Mathematics*, volume 1, page 184). Connected with these functions are certain binary forms of degree equal to one half the totient of n, the divisors of which Professor Lehmer has shown to be of the form $2nx \pm 1$. Combining this result with certain results of Mr. Pierce, Professor Lehmer has also obtained a series of numbers the prime factors of which must belong to two such forms, thus restricting notably the character of their divisors.

7. Professor Haskell shows that the condition that a rational fraction whose denominator is the nth power of a quadratic should be rationally integrable, is that the numerator shall be of degree $2(n - 1)$ and that it shall be apolar to the $(n - 1)$st power of the quadratic factor of the denominator.

THOMAS BUCK,
Secretary of the Section.

TRANSFORMATION THEOREMS IN THE THEORY OF THE LINEAR VECTOR FUNCTION.

BY DR. VINCENT C. POOR.

(Read before the American Mathematical Society, December 31, 1915.)

SINCE the memorable work of Grassmann (1844), the study of the linear transformation has taken various forms, among which are the quaternions of Hamilton, the matrices of Cayley,

the dyadics of Gibbs, and the homography as treated by Burali-Forti and Marcolongo. The notation here followed is that of the book on "Transformations Linéaires" by the last mentioned authors. Reference to the French (1912) edition of this work will be briefly made by the letters B. M. with the section and number following.

The homography is defined by Burali-Forti and Marcolongo* as any linear operator which transforms vectors into vectors. One of the simplest examples of a homography is $\boldsymbol{u}\wedge$, the axial homography,† which transforms all vectors into vectors perpendicular to the vector $\boldsymbol{u}$.

Another concept of fundamental importance in advanced vector analysis is the Grassmann point derivative. If M and P are any two points of space, then $M - P$ is the vector represented by the line segment directed from P to M.‡ If we use $\boldsymbol{u}$ for the difference between any two points of space, and if $f(P)$ is a function depending on the point P, the differential of f, written df, δf, etc., may be defined briefly by the equation

$$df = \lim_{h=0} \frac{f(P + h\boldsymbol{u}) - f(P)}{h}.$$

From this it follows that

$$dP = \boldsymbol{u}$$

or dP is any arbitrary vector. The definition of the point derivative df/dP, following the Leibnitz notation, may now be expressed by the equation

$$\delta f = \frac{df}{dP}\delta P,\S$$

i. e., df/dP is an operator on any arbitrary vector δP which transforms that vector into δf. It may be shown that df/dP is a *linear* operator. In no sense is df/dP to be regarded as a quotient. When the operator df/dP transforms the operand, a vector, into a vector, the point derivative furnishes another example of a homography.

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* B. M., 1.

† The cross $\times$ and the inverted $\vee$ ($\wedge$, read "vec"), standing between two vectors, indicate the scalar and vector products, respectively.

‡ The vector appears as a particular case of the Grassmann "first formation."

§ B. M., p. 60, 5.

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two arbitrary vectors, then the vector of a homography α, designated by $V\alpha$, may be defined as a vector such that

$$2V\alpha \times \boldsymbol{u}\wedge\boldsymbol{v} = \boldsymbol{v}\times\alpha\boldsymbol{u} - \boldsymbol{u}\times\alpha\boldsymbol{v}.^*$$

It is easy to show that

$$\frac{d(\alpha\boldsymbol{u})}{dP}\boldsymbol{x} = \alpha\frac{d\boldsymbol{u}}{dP}\boldsymbol{x} + \left(\frac{d\alpha}{dP}\boldsymbol{x}\right)\boldsymbol{u},^\dagger \tag{1}$$

where α is a homography and $\boldsymbol{u}$ and $\boldsymbol{x}$ are vectors. Transposing the first term of the right member, we have

$$\left(\frac{d\alpha}{dP}\boldsymbol{x}\right)\boldsymbol{u} = \left[\frac{d(\alpha\boldsymbol{u})}{dP} - \alpha\frac{d\boldsymbol{u}}{dP}\right]\boldsymbol{x}.$$

For the brackets in the right member, $S_P(\alpha, \boldsymbol{u})$ will be written. This binary operator is evidently a homography.

The rotational of a homography (written "Rot α"), may be briefly defined as a homography such that

$$(\text{Rot }\alpha)\boldsymbol{u} = 2VS(\alpha, \boldsymbol{u}). \tag{2}$$

The theorems of the present paper are linear transformation theorems which involve the homography, in general, as a function of two points of space. The letter P will be used throughout as the point of integration. It will be understood that the surface σ bounds the region τ and that $\boldsymbol{n}$ is a unit normal at a point P of the surface σ with its positive sense towards the interior of σ.

THEOREM 1. *If α is a homography symmetric in P and M, such that*

$$\frac{d\alpha}{dM} = -\frac{d\alpha}{dP},$$

and if $\boldsymbol{u}$ is independent of M, then

$$2\int V\frac{d(\alpha\boldsymbol{u})}{dM}d\tau = \int \boldsymbol{n}\wedge\alpha\boldsymbol{u}d\sigma + 2\int V\left(\alpha\frac{d\boldsymbol{u}}{dP}\right)d\tau.$$

For, taking $\boldsymbol{x}$ as a function of M alone, and applying (1) and the hypotheses of the theorem, we find that

$$\frac{d(\alpha\boldsymbol{u})}{dM}\boldsymbol{x} = \left(\frac{d\alpha}{dM}\boldsymbol{x}\right)\boldsymbol{u} = -\left(\frac{d(\alpha\boldsymbol{u})}{dP} - \alpha\frac{d\boldsymbol{u}}{dP}\right)\boldsymbol{x}.$$

* B. M., 8 [2].
† B. M., 36 [1].

Remembering the definition of S this result may be written

$$\frac{d(\alpha u)}{dM} = -S_P(\alpha, u)$$

or by using (2), the definition for the rotational of a homography, we have

$$2V\frac{d(\alpha u)}{dM} = -2VS_P(\alpha, u) = -(\mathrm{Rot}_P\,\alpha)u.$$

By substituting this result in the known transformation formula

$$\int(\mathrm{Rot}_P\,\alpha)u\,d\tau = -\int n\wedge\alpha u\,d\sigma - 2\int V\left(\alpha\frac{du}{dP}\right)d\tau,^*$$

the theorem will be obtained. This theorem may be put into a slightly different form if we introduce the definition for the rotational† of a vector, written "rot," which is expressed by the equation

$$\mathrm{rot}_P\,u = 2V\frac{du}{dP}.$$

The theorem will then read

$$\int \mathrm{rot}_M\,\alpha u\,d\tau = \int n\wedge\alpha u\,d\sigma + 2\int V\left(\alpha\frac{du}{dP}\right)d\tau.$$

In expressing the next theorem we will need the conjugate of a homography $K\alpha$, the first invariant of a homography $I_1\alpha$, and the gradient of a homography grad α, of which the gradient of a scalar is a special case. The Maxwell divergence of a vector, which arises, is to be found in any book on the elements of vector analysis. The conjugate of a homography may be defined by the equation

$$K\alpha = \alpha - 2V\alpha\wedge,$$

which is again a homography. The first invariant of a homography α is a scalar, which, for any three arbitrary vectors u, v, w, satisfies the following relation:

$$u\wedge v\times w\cdot I_1\alpha = v\wedge w\times\alpha u + w\wedge u\times\alpha v + u\wedge v\times\alpha w.$$

* B. M., 56 [3], second.

† The rotational of a vector is identical with the Maxwell curl, or the Gibbs $\nabla\times$.

The gradient of a homography α, function of the point P, is a vector such that, for any arbitrary vector $\boldsymbol{u}$,

$$\text{grad}_P\, \alpha \times \boldsymbol{u} = I_1 S_P(K\alpha, \boldsymbol{u}).*$$

THEOREM 2. *If α is a homography function of P and M, such that*

$$\frac{d\alpha}{dM} = -\frac{d\alpha}{dP},$$

and if $\boldsymbol{u}$ is a function independent of M, then

$$\int \text{div}_M\, \alpha\boldsymbol{u} d\tau = \int \boldsymbol{n} \times \alpha\boldsymbol{u} d\sigma + \int I_1\left(\frac{d\boldsymbol{u}}{dP}\alpha\right) d\tau.$$

The proof of this theorem follows easily from the following known theorems, namely:

$$\text{div}_M\, \alpha\boldsymbol{u} = \boldsymbol{u} \times \text{grad}_M\, K\alpha + I_1\left(\alpha\frac{d\boldsymbol{u}}{dM}\right),\dagger$$

which reduces, since $\boldsymbol{u}$ is independent of M, to

$$\text{div}_M\, \alpha\boldsymbol{u} = -\boldsymbol{u} \times \text{grad}_P\, K\alpha,$$

and

$$\int \text{grad}_P\, \alpha \times \boldsymbol{u} d\tau = -\int \boldsymbol{u} \times \alpha\boldsymbol{n} d\sigma - \int I_1\left(\frac{d\boldsymbol{u}}{dP} K\alpha\right) d\tau,\ddagger$$

which may be written

$$\int \boldsymbol{u} \times \text{grad}_P\, K\alpha d\tau = -\int \boldsymbol{u} \times K\alpha\boldsymbol{n} d\sigma - \int I_1\left(\frac{d\boldsymbol{u}}{dP}\alpha\right) d\tau.$$

The substitution of $-\text{div}_M\, \boldsymbol{u}$ for its equal in the left member of the last equation and the application of the commutation theorem

$$\boldsymbol{n} \times \alpha\boldsymbol{u} = \boldsymbol{u} \times K\alpha\boldsymbol{n}$$

lead to the theorem as stated.

The dyad $H(\boldsymbol{u}, \boldsymbol{v})$, another binary operator, is defined by the equation

$$H(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{x} = \boldsymbol{u} \times \boldsymbol{x} \cdot \boldsymbol{v}.$$

The dyad is thus seen to be a homography. We may now express

* B. M., 8 and 37 [3].
† B. M., 41 [3], second.
‡ B. M., 56 [2], first.

THEOREM 3. *If α is a homography, symmetric in P and M, such that*

$$\frac{d\alpha}{dM} = -\frac{d\alpha}{dP},$$

and if $\boldsymbol{u}$ is independent of M, then

$$\int \frac{d(\alpha \boldsymbol{u})}{dM} d\tau = \int H(\boldsymbol{n}, \alpha \boldsymbol{u}) d\sigma + \int \alpha \frac{d\boldsymbol{u}}{dP} d\tau.$$

To demonstrate this theorem we may first observe that

$$\int (\mathrm{div}_P \boldsymbol{v}) \cdot \boldsymbol{u} d\tau = -\int \boldsymbol{v} \times \boldsymbol{n} \cdot \boldsymbol{u} d\sigma - \int \frac{d\boldsymbol{u}}{dP} \boldsymbol{v} d\tau.^*$$

Replacing $\boldsymbol{v}$ by $\boldsymbol{x}$ and $\boldsymbol{u}$ by the vector $\alpha\boldsymbol{u}$, this formula will become

$$\int (\mathrm{div}_P \boldsymbol{x}) \cdot \alpha \boldsymbol{u} d\tau = -\int \boldsymbol{n} \times \boldsymbol{x} \cdot \alpha \boldsymbol{u} d\sigma - \int \frac{d(\alpha \boldsymbol{u})}{dP} d\tau.$$

Under the assumption that $\boldsymbol{x}$ is independent of P this readily reduces to

$$\int S_P(\alpha, \boldsymbol{u}) d\tau = -\int H(\boldsymbol{n}, \alpha \boldsymbol{u}) d\sigma - \int \alpha \frac{d\boldsymbol{u}}{dP} d\tau.$$

With the same restriction on the $\boldsymbol{x}$ we have

$$\frac{d(\alpha \boldsymbol{u})}{dM} \boldsymbol{x} = \left(\frac{d\alpha}{dM} \boldsymbol{x}\right) \boldsymbol{u} = -S_P(\alpha, \boldsymbol{u}) x.$$

From these considerations the theorem will be evident. When the surface integral is identically zero, we have the useful corollary

$$\int \frac{d(\alpha \boldsymbol{u})}{dM} d\tau = \int \alpha \frac{d\boldsymbol{u}}{dP} d\tau.$$

This situation could well happen in the case of an infinite region.

THEOREM 4. *If α is a homography function of P and M and if $\boldsymbol{u}$ is independent of M, and $\boldsymbol{x}$ independent of P, then*

$$\int \frac{d\left(\alpha \dfrac{d\boldsymbol{u}}{dP}\right)}{dM} \boldsymbol{x} d\tau = \int \left(\frac{d\alpha}{dM} \boldsymbol{x}\right) \frac{d\boldsymbol{u}}{dP} d\tau.$$

* B. M., 56 [3], first.

This theorem is demonstrated as soon as the integrands are shown to be equal. Introducing the vector $\boldsymbol{y}$, a function of M alone, then the expression

$$\frac{d\left(\alpha \frac{d\boldsymbol{u}}{dP}\boldsymbol{y}\right)}{dM}\boldsymbol{x}$$

may be written in two different forms,* namely

$$\alpha \frac{d\left(\frac{d\boldsymbol{u}}{dP}\boldsymbol{y}\right)}{dM}\boldsymbol{x} + \left(\frac{d\alpha}{dM}\boldsymbol{x}\right)\frac{d\boldsymbol{u}}{dP}\boldsymbol{y}$$

and

$$\alpha \frac{d\boldsymbol{u}}{dP}\frac{d\boldsymbol{y}}{dM}\boldsymbol{x} + \frac{d\left(\alpha \frac{d\boldsymbol{u}}{dP}\right)}{dM}\boldsymbol{y}.$$

But by the hypotheses of the theorem, the first terms of these expressions may be seen to be equal. The last two are then equal and the theorem follows.

THEOREM 5. *If α is a homography symmetric in P and M and if $\boldsymbol{u}$ is a function independent of M, then*

$$\int \Delta_M'(\alpha\boldsymbol{u})d\tau = -\int\left\{\left(\frac{d\alpha}{dP}\boldsymbol{n}\right)\boldsymbol{u} - \left(\alpha\frac{d\boldsymbol{u}}{dP}\right)\boldsymbol{n}\right\}d\sigma + \int \alpha(\Delta_P'\boldsymbol{u})d\tau.$$

The new symbols involved in this theorem may be defined as follows:

$$\Delta_P'\boldsymbol{u} = \operatorname{grad}\frac{d\boldsymbol{u}}{dP},$$

$$(\Delta_P\alpha)\boldsymbol{u} = \operatorname{grad}\left\{\frac{d(\alpha\boldsymbol{u})}{dP} - 2\alpha\frac{d\boldsymbol{u}}{dP}\right\} + \alpha \operatorname{grad}\frac{d\boldsymbol{u}}{dP}.$$

From these definitions it follows at once that

$$\Delta_P'(\alpha\boldsymbol{u}) = (\Delta_P\alpha)\boldsymbol{u} - \alpha(\Delta_P'\boldsymbol{u}) + 2\operatorname{grad}_P\left(\alpha\frac{d\boldsymbol{u}}{dP}\right).\dagger$$

* Apply equation (1).

† B. M., 50 [3]. The parentheses may be omitted, since Δ operates on homographies only, while Δ' operates only on vectors.

Applying our hypotheses to this equation we have

$$\Delta'_M(\alpha u) = (\Delta_P \alpha)u.$$

This theorem, then, becomes the direct consequence of the formula

$$\int (\Delta_P \alpha)u\,d\tau = -\int \left\{ \left(\frac{d\alpha}{dP} n\right) u - \left(\alpha \frac{du}{dP}\right) n \right\} d\sigma + \int \alpha(\Delta' u)\,d\tau.^*$$

Another form of this theorem is

$$\int \Delta'_M(\alpha u)\,d\tau = \int (\Delta_P \alpha)u\,d\tau,$$

which holds even if the subscript P be replaced by M provided, of course, that u is restricted by the conditions of the theorem.

THEOREM 6. *If* $\alpha \cdot d\beta = d\beta \cdot \alpha$, *then*

$$\int \alpha \text{ grad } \beta\,d\tau = -\int \alpha\beta n\,d\sigma - \int \beta \text{ grad } \alpha\,d\tau.$$

This theorem is easily proved by using the addition theorem for grad $\alpha\beta$ and by applying the "gradient theorem"†

$$\int \text{grad } \alpha\,d\tau = -\int \alpha n\,d\sigma.$$

As a special consequence of this one finds, upon replacing β by the scalar m, that

$$\int \alpha \text{ grad } m\,d\tau = -\int m\alpha n\,d\sigma - \int m \text{ grad } \alpha\,d\tau.$$

The following associated theorem may be proved in a manner similar to that just suggested:

THEOREM 7. *If* $\alpha \cdot d\beta = d\beta \cdot \alpha$ *then*

$$\int (\text{Rot } \alpha)\beta\,d\tau = -\int n \wedge \alpha\beta\,d\sigma - \int (\text{Rot } \beta)\alpha\,d\tau.$$

The purpose of the first five theorems is to transfer the derivative operator from the formal product αu to the homography α or to the vector u alone. The need for such transformations arises naturally in certain studies in applied mathematics.

THE UNIVERSITY OF MICHIGAN,
October 6, 1915.

* B. M., 56 [3], third.
† B. M., 55 [3].

NAPIER'S DESCRIPTIO AND CONSTRUCTIO.

John Napier and the Invention of Logarithms, 1614. A lecture by E. W. HOBSON. Cambridge, University Press, 1914. 48 pp. Cloth, price 1s. 6d.

Napier and the Invention of Logarithms. By G. A. GIBSON. Glasgow, 1914.* 24 pp. Paper.

SOME reference has been already made in this BULLETIN† to the celebration in 1914, under the auspices of the Royal Society of Edinburgh, of the tercentenary of one of the greatest events in the history of science, namely the publication of John Napier's Mirifici Logarithmorum Canonis Descriptio. According to Professor Hobson this work "embodies one of the very greatest scientific discoveries that the world has seen." J. W. L. Glaisher's judgment, which seems to be also that of Professor Gibson (page 13), is, that "with the exception of the Principia of Newton, there is no mathematical work published in the country which has produced such important consequences, or to which so much interest attaches as to Napier's Descriptio."‡

Nor was marked enthusiasm with regard to this work lacking among Napier's contemporaries of eminence. This was especially true of Kepler and Henry Briggs (1556–1630), professor of mathematics in London. In his Ephemeris for 1620 Kepler published as the dedication a letter addressed to Napier, congratulating him warmly on his invention and on the benefit he had conferred upon astronomy. Kepler explains how he verified the canon and found no essential errors in it, beyond a few inaccuracies near the beginning of the quadrant. The letter was written July 28, 1619, about two years after Napier's death, of which Kepler had not heard.

And as to Briggs, he wrote thus to Archbishop Ussher in the year 1615:§ "Napper, lord of Markinston, hath set my

* This pamphlet is a reprint, with separate title-page, of an article in the *Proceedings of the Royal Philosophical Society in Glasgow*. This article is also reprinted in the very interesting and useful volume: Modern Instruments and Methods of Calculation. A Handbook of the Napier Tercentenary Exhibition, edited by E. M. Horsburgh, London [1914], pp. 1–16. For a review of this work by Professor D. E. Smith, see *Science*, n. s., vol. 42, July 23, 1915, pp. 128–129.

† Vol. 21, Dec., 1914, pp. 123–127.

‡ Article "Logarithms" in Encyclopædia Britannica, 11th edition, 1911.

§ Ussher's Letters, 1686, p. 36.

head and hands at work with his new and admireable Logarithms. I hope to see him this summer, if it please God, for I never saw a book which pleased me better and made me more wonder."* Again, Thomas Smith in his Life of Briggs† says of him (Gibson, page 16) when describing his enthusiasm over the Canon Mirificus: "He cherished it as the apple of his eye; it was ever in his bosom or in his hand, or pressed to his heart, and, with greedy eyes and mind absorbed, he read it again and again. . . . It was the theme of his praise in familiar conversation with his friends, and he expounded it to his students in the lecture room."

Briggs paid his anticipated visit to Napier in the summer of 1615, and William Lilly, the astrologer, has told us‡ of his reception at Merchiston: "I will acquaint you with one memorable story, related unto me by John Marr, an excellent mathematician and geometrician, whom I conceive you remember. He was servant to King James I and Charles I. When Merchiston first published his Logarithms Mr. Briggs,

* Sir John Leslie (1766–1832), for many years professor of mathematics and physics in Edinburgh University, was less restrained in his admiration, and characterizes the invention of logarithms as "the noblest conquest ever achieved by man" (Philosophy of Arithmetic, Edinburgh, 1817, p. 156). In a few copies of the English edition of the Constructio published at London in 1616, a cancelled leaf with eulogistic Spencerian verses by one Thomas Bretnor, has not been cut out. These verses begin:

> Pull off your laurel rayes, you learned Greekes
> Let Archimed and Euclid both give way,
> . . . . . .

and the final stanzas are:

> And bonnets vaile; you Germans! Rheticus,
> Reignoldus, Oswald and John Regiomont,
> Lansbergius, Finckius and Copernicus,
> And thou Pitiscus, from whose clearer font
> We sucked have the sweet from Hellespont.
> For were your labours ne'er composed so well
> Great Napier's worth they could not parallel.
>
> By thee great Lord we solve a tedious toyle,
> In resolution of our trinall lines,
> We need not now to carke, to care, or moile,
> Sith from thy witty braine such splendour shines,
> As dazels much the eyes of deepe divines.
> Great the invention, greater is the praise,
> Which thou unto thy nation hence dost raise."

† Vitae quorundam eruditissimorum et illustrium virorum. Londini, 1707.

‡ Lilly's Life, London, 1721.

then reader of the astronomy lectures at Gresham College in London, was so surprised with admiration of them that he could have no quietness in himself until he had seen that noble person whose only invention they were. He acquaints John Marr therewith, who went into Scotland before Mr. Briggs, purposely to be there when these two so learned persons should meet. Mr. Briggs appoints a certain day when to meet at Edinburgh; but, failing thereof, Merchiston was fearful he would not come. It happened one day as John Marr and the Lord Napier were speaking of Mr. Briggs, 'Oh! John,' saith Merchiston, 'Mr. Briggs will not come now'; at the very instant one knocks at the gate, John Marr hastened down and it proved to be Mr. Briggs to his great contentment. He brings Mr. Briggs into my Lord's chamber, where almost one quarter of an hour was spent, each beholding other with admiration, before one word was spoken. At last Mr. Briggs began,—'My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto astronomy, *viz.* the Logarithms; but, my Lord, being by you found out, I wonder nobody else found it out before, when now being known it appears so easy."

Napier and Briggs must have been kindred spirits, for Briggs spent a month at Merchiston, and returned the following summer. He had also planned another visit in the summer of 1617, but Napier's death intervened.

The Descriptio is a small quarto containing an ornamental title page,* preface, 57 pages of explanatory matter, and 90 pages of tables, one page of which is reproduced by Hobson. The preface contains Napier's own account of his invention: "Seeing there is nothing (right well-beloved students of the Mathematics) that is so troublesome to mathematical practice; nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances. And having thought

* A facsimile is given in Mark Napier's Memoirs of John Napier of Merchiston, his lineage, life and times, with a history of the invention of logarithms. London, 1834, p. 374.

upon many things to this purpose, I found at length some excellent brief rules to be treated of (perhaps) hereafter. But amongst all, none more profitable than this which together with the hard and tedious multiplications, divisions and extractions of roots, doth also cast away from the work itself even the very numbers themselves that are to be multiplied, divided and resolved into roots, and putteth other numbers in their place which perform as much as they can do, only by addition and subtraction, division by two or division by three."

The explanatory matter contains "an account of Napier's conception of a logarithm, of the principal properties of logarithms and also of their application in the solution of plane and spherical triangles." Napier's well-known rules of circular parts* containing the complete system of formulas for the solution of right-angled spherical triangles are here given. The tables include the logarithms of the sines of angles from 0° to 90° at intervals of one minute, to seven or eight places of decimals. The arrangement is semiquadrantal, so that the "differentiæ" which are the differences of logarithms in the same line, are the logarithms of the tangents. It should be borne in mind that, "at that time and long afterwards, the sine of an angle was not regarded, as at present, as a ratio but as a length of that semi-chord of a circle of given radius which subtends the angle at the centre. Napier took the radius to consist of 10^7 units, and thus the sine of 90°, called the whole sine, is 10^7; the sines of smaller angles decreasing to zero. The table is therefore one of the logarithms of numbers between 10^7 and 0, not for equidistant numbers, but for the numbers corresponding to equidistant angles." Napier's logarithms are *not* the logarithms now termed Napierian or hyperbolic, that is to say, logarithms to the base e where $e = 2.718\ldots$; the relation between N (a sine) and L its logarithm, as defined in the Descriptio, being $N = 10^7 e^{-L/10^7}$, so that (ignoring the factors 10^7, the effect of which is to render sines and logarithms integral to seven figures), the base is $1/e$. If l (says Glaisher, l. c.) denotes the logarithm to the base e (that is, the so-called "Napierian" or hyperbolic logarithm) and L denotes, as above, Napier's logarithm, the connection between l and L is expressed by

* Cf. I. Todhunter's Spherical Trigonometry revised by Leathem, London, 1901, p. 51 ff. for a discussion of Napier's proof. Reference may also be given to a note by E. O. Lovett in this BULLETIN, vol. 4, July, 1898, pp. 552–554.

$$L = 10^7 \log_e 10^7 - 10^7 l, \quad \text{or} \quad e^l = 10^7 e^{-L/10^7}.$$

In an "admonitio" on the last page Napier states that he will publish the mode of construction of the canon in the Descriptio "si huius inventi usum eruditio gratum fore intellexero." Napier did not live to keep this promise. The proposed work had been written however and was published by his son, Robert Napier (assisted by Henry Briggs), in 1619, under the title: Mirifici logarithmorum canonis constructio. It consists of two pages of preface and 67 pages of text. In the text we find a full account of the method of construction of the canon. There is also here, expressed in words, one of the four formulas for the solution of spherical triangles, known as Napier's analogies; the other three, easily deduced from Napier's result, were formulated by Briggs in the appendix. In the Constructio logarithms are called "artificial numbers" (numeri arteficiales) and Robert Napier states that the work was composed several years before Napier had invented the name logarithm.* The Constructio may therefore have been written many years before the publication of the Descriptio in 1614.

The decimal point in arithmetic appears to have been independently invented by Napier. Decimal fractions were first introduced by Stevin in a tract entitled De Thiende and published in 1585; but his notation is excessively cumbrous. Stevin would have written $652(0)1(1)3(2)7(3)$† or 652137^{0123}, instead of Napier's notation 652·137. It was not till much later, however, that the equivalent of Napier's notation was generally used.

* This term is derived from two Greek words meaning "the number of the ratios." For explanation of the appropriateness of the term consult H. S. Carslaw's paper "The discovery of logarithms by Napier," *Math. Gazette*, vol. 8, p. 82. (The whole paper is given pp. 76–84; 115–119, May, July, 1915. Another paper with the same title was published by Professor Carslaw in *Jour. and Proc. Roy. Soc. New South Wales*, vol. 48, 1914, pp. 42–72.)

It did not take long for the word logarithm to come into English literature. In Act I, scene 1 of Ben Jonson's comedy, "The Magnetic Lady," which was performed in 1632 and first published in 1640, the following lines occur:

> "Sir Interest . . . . . .
> . . . will tell you instantly, by Logarythmes,
> The utmost profit of a stock imployed;"

† A facsimile of a page with this notation from the French edition of Stevin's work is given by D. E. Smith in "History of Decimal Fractions," *Teachers College Bulletin*, (1), no. 5 (March 12, 1910).

There have been several different editions of the Descriptio and Constructio. Of the former in Latin, there were issues in 1614, 1619, 1620, 1658, 1807,* 1857 and 1899; there were English editions in 1616, 1618, 1857. The Latin editions of the latter were in 1619, 1620, 1658 and 1899;† an English edition published by W. R. Macdonald‡ at Edinburgh in 1889 is a scholarly work and contains a professedly complete catalogue of various editions of Napier's works. The bibliophile may find titles supplementary to this catalogue in Quaritch's catalogue no. 336 (March, 1915); good copies of the first English or Latin editions of the Descriptio and of the first Latin edition of the Constructio are worth about $100 apiece.

It is not within the scope of the title which I have chosen for this paper to do more than point out that: (1) it was wholly independent of suggestion from any one else that Napier recognized the advantages, and indicated means for calculation, of a system of logarithms in which $\log 1 = 0$ and $\log 10 = 10^{10}$. "This is practically equivalent to the assumption $\log 10 = 1$, as the former assumption merely indicates that the logarithms are to be calculated to 10 places of decimals." (2) The system of logarithms invented by Joost Bürgi (1552–1632), a Swiss watchmaker and instrument-maker, is decidedly inferior to that of Napier, "and the knowledge of the use of logarithms which was spread in the scientific world was entirely due to Napier."

Both of the sketches under review are remarkably interesting and should be in every mathematical library. Professor Hobson develops, at some length, Napier's methods in connection with his tables.§ This is only briefly touched upon by Professor Gibson who sets forth, in his wonted attractive style, the main facts of Napier's life, dwelling especially upon his intimate relations with Briggs. This latter sketch is the longer although it contains the smaller number of pages.

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* F. Maseres, Scriptores logarithmici. London, vol. 6, 1807.

† Both the Descriptio and Constructio are reprinted by N. W. L. A. Gravelaar in *Verhandelingen der Kon. Akad. van Wet. te Amsterdam*, 1 sectie, deel 6, 1899.

‡ Mr. Macdonald is also the author of the life of Napier in the Dictionary of National Biography.

§ Somewhat extended analysis of Napier's logarithmic works is given by Delambre in his Histoire de l'Astronomie moderne, tome 1, Paris, 1821, pp. 491–506. See also A. von Braunmühl's Vorlesungen über Geschichte der Trigonometrie, zweiter Teil. Leipzig, 1903.

MATHEMATICAL QUOTATION BOOKS.

Memorabilia Mathematica or the Philomath's Quotation-Book. By ROBERT EDOUARD MORITZ. New York, Macmillan, 1914. xiv + 410 pp. Price $3.00.

THE United States has now joined France and Germany in contributing to this class of literature.

It is some twenty-five years since A. M. Rebière published a pioneer work entitled: Mathématiques et mathématiciens. Pensées et curiosités.* The second edition (in later editions there is practically no change) was divided into five parts, headed: Morceaux choisis et pensées (pages 1–178), Variétés et anecdotes (179–340), Paradoxes et singularités (341–470), Problèmes curieux et humoristiques (471–526), Note bibliographique, index and table des matières (527–566).

The textual parts contain, mainly, short quotations or translations (the whole book is in French) from writings ancient and modern. When there is any indication of the source nothing is given, except in rare cases, but the name of the author; the composition of the numerous unsigned paragraphs is attributable to the editor. The work is not, then, strictly a book of quotations but a sort of admixture of quotations, history, mathematical recreations, and table-talk. In defense of his mixture of things gay and serious the author makes appeal to the authority of Pascal: "Les matières de géométrie sont sérieuses d'elles-mêmes, qu'il est avantageux qu'il s'offre quelque occasion pour les rendre un peu divertissantes."

Ahrens's Scherz und Ernst in der Mathematik† differs essentially from the work of Rebière. In the first place the former is strictly a book of quotations; secondly, each quotation is invariably given in the original language, spoken or written; thirdly, exact bibliographical data are provided for all quotations; fourthly, the quotations follow one another consecutively from pages 1 to 495 without grouping under subject headings. A 24-page detailed index of subjects and authors provides the means for rapid orientation. Names of living mathematicians are rarely met with, but references to

* Deuxieme édition, revue et augmentée, Paris, 1893. 2+566 pp.; 4e éd., Paris, 1911.

† W. Ahrens, Scherz und Ernst in der Mathematik. Geflügelte und ungeflügelte Worte. Leipzig, 1904.

the "old masters" such as Abel, Euclid, Euler, Gauss, Helmholtz, Lagrange, Laplace, Steiner, and Weierstrass, are very numerous.

The whole constitutes a most admirable piece of work and must long serve as a desirable model for works of like nature.

The book under review is also a quotation book. The author tells us that ten years were devoted to its preparation and that as a result there have been brought together some 1,200 "more or less familiar passages* pertaining to mathematics, by poets, philosophers, historians, statesmen, scientists, and mathematicians. These have been gathered from over three hundred authors and have been grouped under twenty heads and cross indexed under nearly seven hundred topics." The headings are: Definition and object of mathematics; The nature of mathematics; Estimates of mathematics; The value of mathematics; The teaching of mathematics; Study and research in mathematics; Modern mathematics; The mathematician; Persons and anecdotes; Mathematics as a fine art, as a language; Mathematics and logic, and philosophy, and science; Arithmetic; Algebra; Geometry; The calculus and allied topics; The fundamental concepts of time and space; Paradoxes and curiosities.

Apart from classification in this way the method of Rebière is followed by giving a translation of all quotations in a foreign language, and in probably not more than a score of cases is the original also given. This departure from the scheme of Ahrens must be considered as a great defect, when the book is employed as a work of reference. No scholar is likely to use the translation of a quotation unless he can test its faithfulness by comparison with the original. It is indeed true that exact reference is usually given for this, but then some of the desirable usefulness of the single volume has disappeared.

The compiler has avoided as far as possible traversing the ground that has been trodden already by Rebière and Ahrens. "Thus certain topics, as the correspondence of German and French mathematicians, so excellently treated by Ahrens, have purposely been omitted. The repetitions are limited to a small number of famous utterances whose absence from a work of this kind could scarcely be defended on any grounds."

* This total is about the same as in Ahrens's work. It is no doubt due to the special method of numbering the quotations that some reviewers estimated the total to exceed 2,000; cf. *Mathematical Gazette*, vol. 8 (March, 1915), p. 57; *Nature*, vol. 94 (Oct. 8, 1914), p. 144.

The number of quotations from contemporary or living mathematicians is large. Appended to numerous quotations from the writings of Americans, the names of Bôcher, Cajori, Emerson, Keyser, Myers, B. Peirce, D. E. Smith, W. F. White, E. B. Wilson, J. W. Young, and J. W. A. Young, each of which occurs at least three times, may be noted. John Wesley Young is referred to as *Charles* Wesley Young on page 6. To De Morgan 50 quotations are attributed; to Sylvester 38, to Klein 20, to Shakespeare 3, and so on. As recent biographies have been drawn upon for anecdotes, Sylvanus P. Thompson's Life of Lord Kelvin has not been overlooked. It may be of interest to make a single quotation. First note that elsewhere in this BULLETIN,* I have called attention to the integral $\int_0^\infty e^{-x^2}dx$ which plays an important rôle in the theory of errors, to its connection with certain other integrals studied by Euler and with the curve named by Cesàro the clothoïde, which was considered by Jacob Bernoulli, and by Fresnel in discussing the diffraction of light. The quoted footnote near the end of Thompson's biography is as follows:

"Once when lecturing to a class he [Lord Kelvin] used the word 'mathematician,' and then interrupting himself asked his class: 'Do you know what a mathematician is?' Stepping to the blackboard he wrote upon it:

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}.$$

Then putting his finger on what he had written, he turned to his class and said: 'A mathematician is one to whom *that* is as obvious as that twice two makes four is to you. Liouville was a mathematician.' Then he resumed his lecture."

While Mr. Moritz's work seems to have been very carefully compiled, it is by no means free from at least minor errors (or misprints) and misrepresentations. For example in quotation no. 1043, Mr. Macfarlane is said to have written: "Maxwell denoted Thomson by T and Tait by T': so that it became customary to quote Thomson and Tait's Treatise on Natural Philosophy as T and T'." Instead of T' should of course be T′. In his sketch of Tait, J. S. MacKay refers to "*T and T′* (Thomson et Tait), la notation prolongée T″

* Vol. 20 (June, 1914), pp. 488–489.

servant, chez les amis de Tait, à designer le professeur Tyndall."*

In no. 1049 we read: "His only reply was that he could impossibly interrupt his work"; in 1858 the name of the translator of Dante should be given as Cary; the original of A. C. Orr's mnemonic for π has "In rhymes unapt," not "inapt"; on page 405 of the index for Reid, M. read Reid, T.; all the references after Pope on page 404 are wrong; and on page 410 two quotations are incorrectly attributed to J. W. A. Young instead of J. W. Young.

To state (no. 1007) that Alexander Pope's "Epitaph intended for Sir Isaac Newton" was:

> Nature and Nature's laws lay hid in night:
> God said, "Let Newton be!" and all was light.

is inaccurate. It would have been an easy matter to have turned to the standard edition of Pope's works and found† that the intended epitaph for Westminster Abbey was as follows:

> ISAACUS NEWTONUS
> QUEM IMMORTALEM
> TESTANTUR TEMPUS, NATURA, COELUM:
> MORTALEM
> HOC MARMOR FATETUR
> Nature and Nature's laws lay hid in night:
> God said, *Let Newton be!* and all was light.

Again J. Spence and James Porton are given as authorities for statements (nos. 1023 and 1024) by Newton. Had Sir David Brewster's standard Life of Newton been consulted‡ it would have been found that the first line of 1023 was inaccurate while no. 1024 is made to convey an entirely wrong idea. Instead of "I don't know what I may seem to the world" of the former should be, "I do not know what I may appear to the world." No. 1024 is given as: "If I have seen farther than Descartes, it is by standing on the shoulders of giants." In the course of a letter addressed to Robert Hooke and dated "Cambridge, February 5, 1675–6," occur the sentences: "But, in the mean time, you defer too much to my ability in searching into this subject. What Descartes did was a good step. You have added much several ways, and especially in considering the colours of thin plates. *If I have seen farther, it is by standing on the shoulders of giants.*"

* *L'Enseignement mathématique*, vol. 7, 1905, p. 7.

† Works, edited by Elwin and Courthope, vol. 4, London, 1882, p. 300.

‡ Memoirs of the Life, Writings and Discoveries of Sir Isaac Newton. 2 vols. Edinburgh, 1855, vol. 1, p. 142; vol. 2, p. 407.

But this selection from examples must suffice to illustrate the criticisms made above with regard to errors and careless presentation.

As to additions, I suggest two or three, at random, in connection with things geometrical.

Why not give a reference for Hamilton's "letter to De Morgan (1852)" with regard to the construction of the regular polygon of 17 sides?* And would not the reproduction of Gauss's original announcement of the discovery of the possibility of construction of such a polygon, with ruler and compass,† be worth while?

Why leave out Prior's

> "Circles to square, and cubes to double,
> Would give a man excessive trouble;"‡

And finally, might not the plan of the work permit the inclusion of the verses of the British Museum MS. which shows that Euclid was studied in England as far back as 924–940 A. D.?

> The clerk Euclyde on this wyse hit fonde
> Thys craft of gemetry yn Egypte londe
> Yn Egypte he tawghte hut ful wyde,
> Yn dyvers londe on every syde.
> Mony erys afterwarde y vnderstonde
> Gher that the craft com ynto thys londe.
> Thys craft com ynto England, as y ghow say,
> Yn tyme of good kyng Adelston's day.§

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SHORTER NOTICES.

The Development of Arabic Numerals in Europe. By G. F. Hill. Oxford, Clarendon Press, 1915. 125 pp. Price 7 shillings 6 pence.

It is a commonplace remark that noteworthy achievements in this world often have their inception in the most trivial incidents, and this semiparadoxical law is well illustrated in the work under review. Mr. Hill is the curator of the depart-

* Graves's Life of Sir Wm. R. Hamilton, vol. 3 (1889), pp. 433–435.

† *Intelligenzblatt der allgem. Literatur-Zeitung*, Nr. 66, 1 Junius, 1796, col. 554.

‡ In Alma, canto 3, lines 366–7, published in 1717. Or in Poetical Works of Matthew Prior in 2 volumes, London, 1779, vol. 1, p. 404.

§ J. O. Halliwell, Rara Mathematica, second ed., London, 1841, p. 56.

ment of coins and medals in the British Museum, and a few years ago the date (1481) of an Italian medal of the Sultan Mahomet II, by Costanzo of Ferrara, being called in question, he set about the study of the forms of the so-called Arabic numerals of the early Renaissance period. This naturally led him to extend his researches back to the period of probable introduction of these numerals in Europe, and forward to the time when, owing chiefly to the influence of printing, the forms became practically fixed. It is needless to say that Mr. Hill's search soon carried him beyond the field of numismatics and sigillography, and into the general domain of epigraphy, of medieval manuscripts, and of early typography, so that his essay becomes of value not merely in the field in which he is one of the great living authorities, but in other fields as well.

Such a study has long been needed to assist students in the history of mathematics; for the problem of the date of the introduction of our present numerals into Europe is by no means solved, and aids of this kind are exceedingly valuable. For example, the dates of 800 on the sarcophagus of Pegavus Petrasanta in Milan, of 1007 on a gravestone at Katharein near Troppau, and of 1084 at Castle Acre Priory, are all disconcerting to a beginner in the study of the question, and to have these dates distinctly rejected by a recognized authority on the subject is very helpful. On the positive side, to have at hand a set of tables giving the earliest authentic example of the numerals in the west, the well-known Codex Vigilanus of 976, and a score of other examples of the Boethian apices; to have the most characteristic specimens of the Arabic forms from the twelfth to the sixteenth century, not only as shown in the manuscripts, but also in the inscriptions and on the paintings of these periods; to be able at a glance to follow the lambda seven in its efforts to assume the upright form; to see the looped four giving place to the familiar form which finally replaced it; to see the beginning of the struggle of the zero for a place and for a recognized shape,—to have all this in scientific tabular form so that the changes can be traced at a glance, is to visit what may be described quite seriously as the cinematograph of our numeral system.

The tables, sixty-four in number and with hundreds of carefully drawn examples, are not without their surprises to those who have not looked into the subject. For example, the earliest known manuscript with the apices gives the upright

form of the seven instead of the lambda form which is commonly supposed to be the earlier one, and this is true, it may be said in passing, not only of the manuscript from which Mr. Hill has taken his illustration, but also the other Escorial manuscript of the same work. Likewise the upright four, which we ordinarily think of as due to the Florentines of the fifteenth century, who indeed had much to do with establishing it, is shown to have been used in the thirteenth and fourteenth centuries by English scribes and early in the fourteenth century by the Italians, probably Florentine monks, and quite commonly in the fifteenth century by writers of English manuscripts.

What strikes the reader as most gratifying is that Mr. Hill has brought to the problem a perfectly judicial mind; he has no thesis to defend; he is advocate for no party to any controversy; he is the scholar seeking absolute truth. To his researches, to his patience, to his care in weighing evidence, all who have an interest in the history of mathematics are quite as much his debtors as those whose fields of interest are in the lines of numismatics and paleography.

DAVID EUGENE SMITH.

Introduction to Infinitesimal Calculus. By G. W. CAUNT. Oxford, The Clarendon Press, 1914. xx+568 pp.

THIS book is an attempt to present the calculus in a way that will appeal to students of engineering. The author expresses a hope that he has made the book rigorous enough to satisfy the instructor in a first course in calculus for a student in pure mathematics. This seems to be rather an exception, most texts being written for the pure mathematician, or at least chiefly from his viewpoint. The subject matter is that usually found in the texts on calculus with the addition of a chapter on differential equations, and the author presents the subject from the viewpoint of the engineer. The book is written for a first course in calculus and is arranged for a minimum amount of analytic geometry to precede it. The author usually introduces a subject by means of a number of illustrative numerical examples worked out in detail, thus leading the student into a subject by means of his interest in the purpose it serves. This use of numerical examples, completely solved out, prepares the student of engineering to make use of his mathematics in his engineering courses. Too often a

student gets through his calculus without being able to apply it to numerical problems, especially when they occur in a course not designated as mathematics. This should not be so, and is undoubtedly the reason why mathematics in our technical schools is in such disfavor with the students. This situation is most probably due partly to the mathematics teachers, partly to the engineering teachers and partly to the textbooks. Wherever the fault lies, any textbook written for engineering students should bridge this gap or wholly close it. If there is any essential difference between the calculus for the engineer and the calculus for the pure mathematician, then our textbooks for engineers should be written as such and should not be attempts to compile books that can be sold to both classes of students. The teacher of mathematics in an engineering school who is seeking to present the calculus to his students in a way that will make it appeal to them as being a subject they need instead of one they must take will find this book a help in that direction.

The first two chapters, especially the second on "Limits and continuous functions," are an attempt to get the student familiar with subjects that often remain hazy until the end of his course in calculus. They present the matter in a very clear way by means of many examples with full explanations. The remaining chapters are treated in much the same way. The book is so arranged that a shorter course can be had by omitting certain chapters without destroying the continuity of presentation. The book contains more material than most of our engineering schools could cover in the time now allotted to mathematics. The author seems to have had liberty from his publishers to give as much space as he desired to illustrative problems and lists of exercises. This is a very good feature of the book. On the whole the book should prove very teachable.

T. E. MASON.

Die Rechenmaschinen und das Maschinenrechnen. Von Dipl. Ing. LENZ. Band 490, Sammlung aus Natur und Geisteswelt. Leipzig, Teubner, 1915. vi+114 pp.

OVER 500 separate numbers of this collection of booklets on science, the arts, and technology have been published and the set is not yet closed. Each volume is complete in itself and retails at M. 1.25.

No. 490—the one under review—aims to give its readers

clear and definite notions concerning the mathematical and mechanical principles underlying the construction of the many types of machines designed to perform automatically the operations of addition, subtraction, multiplication, and division.

No great mathematical or technical knowledge is required to read the book with ease; though it possesses much more of scientific interest and spirit than one might expect to find in a so-called "popular" treatise.

Representative machines, mostly of German or American make, varying in complexity from the abacus to the Burroughs —all designed to add or subtract—are described in detail, and the mechanical principles according to which they operate are discussed. Similarly there are separate chapters on machines designed to perform multiplication and division, the highest type of which is represented by the "millionaire" computing machine so often found in our mathematical laboratories. The text is illustrated by 45 excellent figures.

The author summarizes the present state of development to which mechanical computation has been carried; points out many imperfections which still exist, and suggests the requirements which the ideal machine should fulfill. He closes with a rather brief discussion of the principles underlying the construction and use of the slide rule. It seems to the reviewer that this chapter is rather inadequate and hardly up-to-date.

ERNEST W. PONZER.

Die mathematische Ausbildung der Deutschen Landmesser. Von PH. FURTWÄNGLER und G. RUHM. Band IV, Heft 8, I. M. U. K. Leipzig, Teubner, 1914. vi+50 pp.

IN this pamphlet is given a summary of the training, both practical and theoretical, together with the courses of study prescribed for the German engineer who wishes to specialize in land surveying. Though the various German states differ in their minimum requirements for the holder of this office, who must pass a rigid examination, yet nowhere are there evidences of the existence of an elective sinecure such as is represented by that of our own county surveyor, an office too often filled by some derelict engineer with a political pull. The work is systematized and is more of the grade of that of our Coast and Geodetic Survey.

Special courses are offered at the technical high schools in

the different states for the training of this class of engineers. The authors go into these in detail. A composite cross-sectional view of present practise would show about the following.

At least one or two years of practical training under the guidance of a regularly appointed surveyor. This practical experience may precede or follow the course of instruction in the technical schools. An average of about four semesters in a technical school offering the special courses required. An opportunity to advance in the profession. Advancement based solely on merit.

The courses offered necessarily include trigonometry, algebraic analysis, analytic and descriptive geometry, the calculus, mechanics, sometimes the method of least squares, astronomy, geodesy, drafting, map-making, seminar.

A general survey is made of the methods of handling these various subjects. And one is not surprised to find that throughout there are emphasized all those which aim to develop the order, accuracy, simplicity, and efficiency so desirable in any engineer.

Among suggestions intended to secure a greater efficiency the authors include such as the founding of higher schools, a three-year course of study, and a longer apprenticeship.

ERNEST W. PONZER.

Algebraic Invariants. By LEONARD EUGENE DICKSON. (No. 14, Mathematical Monographs, edited by Mansfield Merriman and Robert S. Woodward.) New York, John Wiley and Sons, 1914. x+100 pp.

IN this brief introduction to the classical theory of invariants Professor Dickson puts the reader under a further debt of gratitude for the excellent and entertaining way in which he is led to a first acquaintance with the important subject of invariants. It is difficult to conceive how one could be more comfortably drawn into a knowledge of invariants and covariants than by the gradual and lucid processes of the early part of this book. In the first ten pages there is a progressive approach to the full notion of invariant, carried forward from things well known to the beginner by means of processes and ideas which are of intrinsic interest and value in themselves. After a similar gradual development of the notion of covariant the formal definitions of invariants and covariants are given on pages 14 and 15.

The book falls into three parts of nearly equal length. Part I (pages 1–29) treats of linear transformations both from the standpoint of a change of points of reference and from the standpoint of projective geometry. Its purpose is to lead the reader by easy stages into a proper conception of the nature of the subject. How this purpose is achieved may be seen in part by the way in which the notions of invariant and covariant are introduced and illustrated by means of the simplest examples. Certain covariants such as Jacobians and Hessians are discussed and their algebraic and geometrical interpretations are given. In particular, the Hessian is employed in the solution of the cubic equation and in the discussion of the points of inflection of the plane cubic curve.

In connection with the interesting illustrative examples and applications of Part I the author finds opportunity to derive or illustrate several general elementary theorems so that the reader is making substantial progress in the theory as he learns its meaning through examples.

Part II (pages 30–62) is devoted to a systematic development, in non-symbolic notation, of the algebraic properties of invariants and covariants, chiefly of binary forms. In the preceding part the reader has been prepared so as to be able to follow this treatment with comfort. There is here a compact but lucid treatment of the following topics: homogeneity, weight, annihilators, alternants, seminvariant leaders of covariants, number of linearly independent seminvariants, Hermite's law of reciprocity, fundamental systems of covariants, canonical form of the binary quartic, properties of invariants and covariants as functions of the roots, and differential operators producing covariants. Moreover, irrational invariants are illustrated (in § 35) by a carefully selected set of exercises.

In part III (pages 63–97) is given an introduction to the symbolic notation of Aronhold and Clebsch. This notation is first explained at length for a simple case and the reader is led gradually into its use through a carefully constructed proof of the fundamental theorem on the types of symbolic factors of a term of a covariant of binary forms. That a fundamental system of covariants is finite is proved (pages 70–76) by a method due to Hilbert. The theory of transvectants is developed so far as needed in the treatment of apolarity and its application to rational curves and in the inductive deter-

mination of fundamental systems of covariants. Finally, there is a discussion of the concomitants of ternary forms in symbolic notation. R. D. CARMICHAEL.

Leçons sur la Théorie des Fonctions. Par EMILE BOREL. Deuxième édition. Paris, Gauthier-Villars, 1914. xi+259 pp.

FOR purposes of review the second edition of this valuable book by Borel may be divided into three parts: the body of the text exclusive of the extensive notes at the end (Chapters I to VI, pages 1–101); notes contained in the first edition (pages 102–134); notes added in the second edition (pages 135–256). The first two parts are reprinted without modification except for a single change in a matter of terminology in the theory of point sets. Since these parts have been before the mathematical public for many years (having been originally published in 1898), they call for no further review now. The third part consists of three notes numbered IV, V, VI; it makes up about half of the present volume.

Note IV (pages 135–181) is devoted to polemics concerning the transfinite and the demonstration of Zermelo. It contains a reprint of seven articles, principally by Borel, published from time to time during the years 1899 to 1914. These discussions are perhaps of more interest to philosophers than to mathematicians. A perusal of this note in comparison with earlier statements by Borel shows that his thought on some of the matters in consideration has undergone a marked evolution.

Note V (pages 182–216) is devoted to denumerable probabilities and their arithmetic applications.

Finally, Note VI (pages 217–256) is given to a development of the theory of measure and of integration from the point of view adopted by Borel in his definition of measurable sets. It contains the most important matter added in this new edition of the work. The subject is approached in a very elementary and simple way and the treatment is carried through to results of considerable generality both in the theory of measure and in the theory of integration. The treatment will serve conveniently as an introduction to the fundamental researches of Lebesgue in the theory of integration.

R. D. CARMICHAEL.

Radioactive Substances and their Radiations. By E. RUTHERFORD, D.Sc., Ph.D., LL.D., F.R.S. Cambridge University Press, 1913. viii+699 pp.

THIS book contains a very interesting treatment of a fascinating subject by a writer who speaks with the authority of a leading investigator. Dealing effectively as it does with one of the most remarkable developments of modern physics, a science which has sprung anew into the most rapid growth in the present generation, it is assured of a wide range of readers and of a valuable place in the development of science. For the most part it deals with experimental researches and results and makes but a small use of mathematics in their interpretation. For this reason only a brief notice of it should be given here, although it appears to be a book of more than usual value.

The investigations of radioactivity are too new for the subject yet to have taken on a mathematical form. Mathematics is the dress in which a physical science is best expounded and developed after it has come to maturity and the fundamental laws involved in it have been determined with precision; but it ill becomes a subject while yet in the infancy of early experimental stages.

But in spite of this newness of the subject there is one mathematical investigation connected with it and of considerable interest owing to the differential equation to which it gives rise. This seems not to be mentioned by Rutherford. It was initiated by J. J. Thomson (see Encyclopaedia Britannica, 11th edition, volume 6, page 371) and further developed by Mie (*Annalen der Physik* (4) 13 (1904): 857–889) and others. In investigating the distribution of electric force when a current is passing through an ionized gas Thomson obtained a differential equation which Mie has transformed to

$$\frac{1-k^2}{2\lambda}z\frac{d^2z}{d\xi^2}-\frac{1-k^2}{4\lambda^2}\left(\frac{dz}{d\xi}\right)^2+\frac{k}{\lambda}\frac{dz}{d\xi}-z+1=0,$$

where k and λ are constants. In developing the theory of this equation considerable difficulty has arisen. A reading of the papers which have been devoted to it leaves one with the conviction that they do not satisfactorily dispose either of the mathematical question involved in the solution of the equation or of the electrical question of which this equation is in part the mathematical expression.

R. D. CARMICHAEL.

The Mathematical Analysis of Electrical and Optical Wave-Motion on the Basis of Maxwell's Equations. By H. BATEMAN, M.A., Ph.D. Cambridge University Press, 1915. vi +159 pp.

THIS book is intended as an introduction to certain recent developments of Maxwell's electromagnetic theory which are directly connected with the solution of the partial differential equation of wave motion. The higher parts of the theory which are based on the dynamical equations of motion are not considered.

In Chapter I (pages 1–24) the author starts from the fundamental equations for free ether in Maxwell's electromagnetic theory and shows in the first place that solutions of these equations may be obtained by means of solutions of the fundamental wave equation

$$(1) \qquad \Omega(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,$$

where c is the velocity of light (and is taken to be a constant). A function u of x, y, z, t, satisfying equation (1), is called a wave function. In the second place the author exhibits a class of solutions of the fundamental equations of Maxwell by means of functions α and β (of x, y, z, t) of such a nature that if $F(\alpha, \beta)$ is an arbitrary function of α and β, F satisfies the partial differential equation

$$(2) \qquad \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2 = \frac{1}{c^2}\left(\frac{\partial F}{\partial t}\right)^2.$$

Continuing in Chapter I it is shown that the fundamental Maxwell equations for a material medium may be solved by means of functions u (of x, y, z) satisfying the equation

$$(3) \qquad \Delta u + k^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0,$$

the quantity k being a constant with respect to x, y, z, t. This equation is obviously satisfied by wave functions of the form

$$u = e^{\pm ikct} f(x, y, z).$$

In connection with a wave boundary whose equation may be expressed in the form $F(x, y, z, t) = 0$ equation (2) comes again into play and here expresses the fact that the moving wave

boundary moves normally to itself with the velocity of light.

The work under review is devoted principally to deriving the properties of functions satisfying equations (1) and (3) and to the applications of the resulting theory to problems of electrical and optical wave motion. It is desirable that all types of functions satisfying equations (1) and (3) should be studied and not merely those which admit readily of application to physical problems; for in this way a clearer light is thrown on the physical problems themselves. But there is a much more important reason why the scope of the inquiry should not be restricted. The theory of wave functions forms a natural extension of the theory of functions of a complex variable and may consequently lead to results of great value for the general theory of functions. Moreover, the theory of functions of two complex variables is closely connected with the theory of wave functions. Our author recognizes this breadth of range of interest in the theory of equations (1) and (3) and takes account of it in his exposition.

In Chapter II (pages 25–34) is to be found a general survey of the different methods of solving the wave equation, including the Bernoulli method of reduction to ordinary differential equations, the generalization of wave functions through the property of linearity of the wave equation and the method of transformations (developed by Bateman) by means of which other solutions may be obtained from a single given solution.

By means of a transformation to polar coordinates and a use of the Bernoulli method of reduction to ordinary differential equations various solutions of equations (1) and (3) are obtained in Chapter III (pages 35–68), and these are employed in the investigation of several important problems connected with spherical obstacles, including the following: scattering of electromagnetic waves by spherical obstacles; damped vibrations for the space outside of a sphere; polarization and intensity of the scattered light; absorption of light by a spherical obstacle; pressure of radiation on a spherical obstacle.

In Chapter IV (pages 69–81) a similar treatment is made of equations (1) and (3) by means of a transformation to cylindrical coordinates and the results are applied in a treatment of the propagation of waves on a semi-infinite solid bounded by a plane surface and also along a straight wire of circular cross section.

A partial solution of the problem of diffraction is obtained in Chapter V (pages 82–94) by means of multiple-valued solutions of the wave equation.

Chapter VI (pages 95–109) contains a brief account of each of several transformations of coordinates which lead to solutions of the wave equation suitable for the treatment of problems connected with surfaces of revolution. These give rise to several important ordinary differential equations of which it is desirable to have a more complete theory than we possess at present.

In Chapter VII (pages 110–114) methods are exhibited for finding homogeneous solutions of the wave equation.

Chapter VIII (pages 115–140) is given to a treatment of electromagnetic fields with moving singularities. It consists principally of the author's own contributions to the theory in consideration. From the investigations of this chapter it is seen that the mathematical analysis connected with equations (1) and (3) is suitable for the discussion of three distinct theories of the universe which may be briefly described as follows: (*a*) the ether is a continuous medium and matter consists of aggregates of discrete particles; (*b*) the ether is a discontinuous medium consisting of a collection of tubes or filaments; and matter is an aggregate of discrete particles attached to the tubes; (*c*) the ether is a continuous medium and matter is an aggregate of discrete particles to which tubes are attached. In Chapters I to VII the analysis is adapted almost entirely to the first of these theories, the high development of which we owe to the pioneer work of Maxwell, Fitzgerald, Hertz, Rayleigh, Heaviside, J. J. Thomson, Lorentz and Larmor. The other theories have heretofore not received much attention; and hence the contributions of Bateman (in Chapter VIII) form a welcome addition to our previous knowledge. It is to be hoped that they will lead to further developments so that a comparison can be made between the different theories. It is likely that each of the theories will be enriched by the development of the other two.

Finally, some miscellaneous theories are briefly treated in the concluding Chapter IX (pages 141–154).

This book will be found valuable for its new contributions to the theory of equations (1) and (3), for its account of the present state of knowledge and its numerous references to the literature of these equations and for its indication of several

points at which further development of the theory is desirable. Its greatest value probably lies in the way in which it makes clear that it is desirable to have a further and deeper investigation of the properties of wave functions and the wave equation. R. D. CARMICHAEL.

NOTES.

THE Swiss mathematical society held its annual meeting at Geneva, September 12–15, in affiliation with the centenary celebration of the Helvetian society of natural scientists. The following papers were read: By Professor L. G. DU PASQUIER, "On systems of complex numbers"; by Dr. G. PÓLYA, "Is the non-continuation of a power series the general case?"; by Professor M. PLANCHEREL, "On the convergence of a remarkable class of definite integrals containing an arbitrary function"; by Dr. W. H. YOUNG and Dr. G. C. YOUNG, "Integration with regard to a function of limited variation"; by Dr. G. C. YOUNG, "On curves without tangents"; by Dr. K. MIRIMANOFF and Dr. G. C. YOUNG, "On the theorem of 'tuiles'"; by Professor L. J. CRELIER, "On a particular theorem of the geometry of motion"; by Professor R. DE SAUSSURE, "Geometry of leaflets"; by Professor CAILLER, "On the analytic theory of directed bodies"; by Dr. H. BERLINER, "A new projective analytic geometry"; by Professor L. KOLLROSS, "Concerning a duality"; by Dr. F. GONSETH, "Generalization of a theorem of Poncelet"; by E. GUILLAUME, "On the impossibility of reducing the law of divergence in several variables to a composite probability."

Professor M. GROSSMANN, of the technical school at Zurich, was elected president for the following year.

AT a meeting of the Edinburgh mathematical society on November 12 the following papers were read: "The solution of difference equations by continued fractions," by J. A. STRANG; "A suggested measurement of relationship" and "The equation $x^3 + y^3 + z^3 + u^3 = 0$ where x, y, z, u are rational," by J. E. A. STEGGALL; "Notes on a triangle," by G. E. CRAWFORD; "Easy geometrical proof of a theorem of Chasles," by E. PRESS.

THE Association of mathematics teachers of New Jersey held its third regular meeting at Stevens institute of technology on November 20. The programme included the following papers: "Mathematics and insurance," by P. C. H. PAPPS; "The proper functioning of a high school course in geometry," by R. T. LEVALLEY; "Review of Bourlet's Plane Geometry," by B. B. STRANG; "A high school course in strength of materials," by G. D. ORNER. Professor H. B. FINE is president of the association.

THE December number (volume 17, number 2) of the *Annals of Mathematics* contains the following papers: "The Cauchy definition of a definite integral," by D. C. GILLESPIE; "Surfaces with isothermal representation of their lines of curvature as envelopes of rolling," by L. P. EISENHART; "A theorem concerning real functions," by K. P. WILLIAMS; "Note on an operation of the third grade," by A. A. BENNETT; "Determination of all triply orthogonal systems containing a family of minimal surfaces," by T. H. GRONWALL.

THE May number of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers: "A new canonical form of the elliptic integral," by BESSIE I. MILLER; "Transformations of conjugate systems with equal invariants," by L. P. EISENHART. The June number of the *Proceedings* contains: "Solution of an infinite system of differential equations of the analytic type," by F. R. MOULTON; "On the factorization of various types of expressions," by HENRY BLUMBERG. The July number contains: "On the representation of arbitrary functions by definite integrals," by W. B. FORD; "Some theorems connected with irrational numbers," by W. D. MACMILLAN. The August number contains "Seven points on a twisted cubic curve," by H. S. WHITE, and the October number contains "On isothermally conjugate nets of space curves," by G. M. GREEN.

THE Macmillan Company announces the publication of an Historical Introduction to Mathematical Literature, by G. A. MILLER.

THE firm of B. G. Teubner in Leipzig announces that the following works are in press: Euler's works, series 1, volume

2, part 1: Commentationes arithmeticae, by F. RUDIO; volume 18, part 2: Commentationes analyticae ad theoriam integralium pertinentes, by A. GUTZMER—Lie's collected memoirs, volume 3, by F. ENGEL—Vorlesungen über Zahlen- und Funktionenlehre, by A. PRINGSHEIM—The second part of the second volume of Darstellende Geometrie, by E. MÜLLER—Theorie der elliptischen Funktionen, by R. FRICKE —Vorlesungen über reelle Funktionen, by C. CARATHÉODORY —Differential- und Integralrechnung, by L. BIEBERBACH— Les théories statistiques en thermodynamique, by H. A. LORENTZ—Partial differential equations of mathematical physics, by A. G. WEBSTER—and a number of reports on the teaching of mathematics in Germany, edited by F. KLEIN.

THE following courses in mathematics are given during the present winter semester:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Analytic geometry, four hours; Synthetic geometry, four hours; Elementary derivation of the most important properties of conics, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Theory of curves and surfaces, four hours; General problems of the theory of functions, two hours; Seminar, two hours.—By Dr. K. KNOPP: Ordinary differential equations, II, four hours; Theory of functions, II, with exercises, five hours.—By Professor R. ROTHE: Integral calculus, four hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Analytic geometry of two and of three dimensions, four hours; Seminar, two hours.—By Professor F. LONDON: Differential and integral calculus, with exercises, five hours; Synthetic geometry, with exercises, three hours; Seminar, two hours.—By Professor I. SCHUR: Selected chapters of analysis, two hours; Introduction to the theory of functions, four hours; Seminar, two hours.—By Dr. J. O. MÜLLER: Calculus of variations, three hours; Seminar, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Development of mathematics in the nineteenth century, two hours.—By Professor D. HILBERT: Differential equations, four hours; Seminar, two hours.—By Professor C. RUNGE: Descriptive geometry, with exercises, five hours; Differential

and integral calculus, with exercises, six hours; Seminar, two hours.—By Professor E. LANDAU: Algebra, four hours; Seminar, two hours.—By Professor C. CARATHÉODORY: Curves and surfaces, four hours; Selected chapters from the theory of functions, with exercises, four hours.—By Professor F. BERNSTEIN: Mathematical statistics, two hours; Mathematics of insurance, two hours; Seminar, two hours.

UNIVERSITY OF LEIPZIG.—By Professor O. HÖLDER: Differential and integral calculus, five hours; Elliptic modular functions, two hours; Seminar, two hours.—By Professor K. ROHN: Analytic geometry of space, four hours; Differential geometry, four hours; Seminar, two hours.—By Professor G. HERGLOTZ: Mechanics, five hours; Selected chapters from the theory of numbers, three hours; Seminar, two hours.—By Dr. K. BLASCHKE: Partial differential equations, four hours; Conformal representation, two hours; Seminar, two hours.

UNIVERSITY OF STRASSBURG.—By Professor F. SCHUR: Analytic geometry of two and three dimensions, four hours; Selected chapters of differential geometry, two hours; Seminar, two hours.—By Professor G. FABER: Differential and integral calculus, four hours; Elliptic functions, two hours; Seminar, two hours.—By Professor M. SIMON: History of mathematics in ancient times, two hours.—By Professor J. WELLSTEIN: Graphical statics, three hours.—By Professor P. EPSTEIN: Analytic treatment of projective geometry, two hours.—By Dr. A. SPEISER: Perspective, two hours.

DR. J. RADON and Dr. F. RULF have been appointed docents in mathematics at the technical school of Vienna.

DR. W. SIMANDL has been appointed docent in projective geometry at the Bohemian technical school of Brünn.

DR. K. BOPP, of the University of Heidelberg, has been promoted to an associate professorship of mathematics.

PROFESSOR G. FABER, of the University of Strassburg, has been appointed to a full professorship in the technical school at Munich.

PROFESSOR P. PAINLEVÉ, of the University of Paris, has been appointed minister of education in the new French cabinet.

PROFESSOR E. PASCAL, of the University of Naples, has been awarded the medal of the Italian society of sciences (the forty).

PROFESSOR J. MOLLERUP has been appointed associate professor of mathematics in the University of Copenhagen, succeeding Professor H. BOHR, who has been called to the Copenhagen technical school as successor of the late Professor P. C. V. HANSEN.

DR. G. ARMELLINI, of the University of Rome, has been appointed associate professor of mechanics at the technical school of Turin.

PROFESSOR M. CIPOLLA, of the University of Catania, has been promoted to a full professorship of algebraic analysis.

DR. L. L. RIOS has been appointed docent in mechanics at the University of Padua.

THE two Benjamin Peirce instructorships in mathematics at Harvard University (see BULLETIN, volume 21, page 315) are again open to general competition. Applications for the year 1916–17, accompanied by the necessary papers, should reach Professor BÔCHER not later than February 1, 1916, who will be glad to furnish further particulars.

PROFESSOR N. C. GRIMES, formerly of the University of Arizona, has been appointed to a professorship of mathematics in the University of Oregon.

MISS GERTRUDE I. MCCAIN has been appointed professor of mathematics in Oxford College, not in the Western College for Women as erroneously announced in the December BULLETIN. The professorship of mathematics in the Western College for Women is held by Miss HARRIET E. GLAZIER.

AT the State University of Kentucky associate professor J. M. DAVIS has been promoted to a full professorship of mathematics.

MISS A. D. LEWIS, of Mt. Holyoke College, has been appointed professor of mathematics and head of the department at the Kentucky College for Women.

At the Iowa State College the following changes have been made in the department of mathematics: Miss A. Fleming has been promoted to an assistant professorship; Mr. J. R. Sage, Miss F. Farnum, and Miss M. Miller have been appointed instructors; Miss N. Madson has been appointed assistant.

Professor H. L. Rietz, of the University of Illinois, has been appointed by Governor Dunne a member of the committee to investigate the operation of national and foreign pension systems.

Professor W. H. H. Hudson, of King's College, London, died September 21, at the age of 76 years.

Professor E. Janisch, of the technical school at Prague, died August 11, 1915, at the age of 46 years.

Professor J. Knoblauch, of the University of Berlin, died July 22, 1915, at the age of 59 years.

Professor F. Leconte, of the University of Ghent, died October 11, 1915, at the age of 50 years.

Professor H. Ganter, of the cantonal school at Aargau, died July 29, 1915, at the age of sixty-seven years.

Dr. L. Orlando, of the University of Rome, was killed in battle in August.

Dr. R. Torelli, of the University of Pisa, was killed in battle August 10, 1915, at the age of thirty-one years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Berzolari (L.). See Encyclopédie.

Besserve (A.). Le cercle et les surfaces cerclées en géométrie conforme. (Thèse.) Paris, Gauthier-Villars, 1915. 4to. 8+162 pp. Fr. 9.00

Bôcher (M.). Plane analytic geometry. New York, Holt, 1915. 12mo. 13+235 pp. $1.60

Brown (E. H.). See Burn (J.).

Burn (J.) and Brown (E. H.). Elements of finite differences; also solutions to questions set for part 1 of the examinations of the Institute of actuaries. London, Layton, 1915. 10s. 6d.

CARRUS (S.). See ENCYCLOPÉDIE.

CARSE (G. A.) and SHEARER (G.). A course in Fourier's analysis and periodogram analysis for the mathematical laboratory. (Edinburgh Mathematical Tracts No. 4.) London, Bell, 1915. 8vo. 8+66 pp. 3s. 6d.

CARTAN (E.). See ENCYCLOPÉDIE.

CHARLES (R. L.). See FRANKLIN (W. S.).

DINGELDEY (F.). See ENCYCLOPÉDIE.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome III, volume 1, fascicule 2: Les notions de ligne et de surface par H. von Mangoldt et L. Zoretti; Exposé parallèle du développement de la géométrie synthétique et de la géométrie analytique pendant le 19ième siècle par G. Fano et S. Carrus; Géométrie énumérative par H. G. Zeuthen et M. Pieri; La théorie des groupes continus et la géométrie par G. Fano et E. Cartan. Paris, Gauthier-Villars et Leipzig, Teubner, 1915. Gr. 8vo. Pp. 161–352. M. 7.20

——. Tome III, volume 3, fascicule 2: Systèmes de coniques par F. Dingeldey et E. Fabry; Théorie générale des courbes planes algébriques par L. Berzolari. Paris, Gauthier-Villars et Leipzig, Teubner, 1915. Gr. 8vo. Pp. 161–304. M. 5.40

FABRY (E.). See ENCYCLOPÉDIE.

FANO (G.). See ENCYCLOPÉDIE.

FORD (L. R.). An introduction to the theory of automorphic functions. (Edinburgh Mathematical Tracts No. 6.) 8vo. 8+96 pp. 3s. 6d.

FRAENKEL (A.). Ueber die Teiler der Null und die Zerlegung von Ringen. (Dissertation.) Marburg, 1914.

FRANKLIN (F. W.). Thoughts on ultimate problems. London, David Nutt, 1915. 150 pp. 1s. 6d.

FRANKLIN (W. S.), MACNUTT (B.) and CHARLES (R. L.). Calculus supplement. To take the place of pages 1–41 of the 1913 edition. South Bethlehem, Pa., New Era Printing Co., 1915. 8vo. 51 pp. Paper.

GIBB (D.). A course in interpolation and numerical integration for the mathematical laboratory. (Edinburgh Mathematical Tracts No. 2.) London, Bell, 1915. 8vo. 8+90 pp. 3s. 6d.

GLENN (O. E.). Treatise on the theory of invariants. Boston, Ginn, 1915. 8vo. 10+245 pp. Cloth. $2.75

KAMPÉ DE FÉRIET (J.). Sur les fonctions hypersphériques. (Thèse.) Paris, Gauthier-Villars, 1915. 4to. 112 pp. Fr. 6.00

KEYSER (C. J.). Science and religion: the rational and the supernatural. Address delivered before the Phi Beta Kappa Association in the City of New York. New Haven, Yale University Press, 1914. 12mo. 75 pp. $0.75

——. The new infinite and the old theology. New Haven, Yale University Press, 1915. 12mo. 117 pp. $0.75

KÖNNEMANN (W.). Rationale Lösungen von Aufgaben aus dem Gebiete der gesamten Elementarmathematik in funktionaler Abhängigkeit. Berlin, Winckelmann, 1915. 112 pp. M. 3.00

MACNUTT (B.). See FRANKLIN (W. S.).

MANGOLDT (H. VON). See ENCYCLOPÉDIE.

PÉRÈS (J.). Sur les fonctions permutables de première espèce de M. Vito Volterra. (Thèse.) Paris, Gauthier-Villars, 1915. 4to. 100 pp. Fr. 5.00

PIERI (M.). See ENCYCLOPÉDIE.

PRANGE (G.). Die Hamilton-Jacobische Theorie für Doppelintegrale (mit einer Uebersicht der Theorie für einfache Integrale). (Dissertation.) Göttingen, 1915.

SHEARER (G.). See CARSE (G. A.).

SCHMIDT (M. C. P.). Kulturhistorische Beiträge zur Kenntnis des römischen Altertums. 1tes Heft: Zur Entstehung und Terminologie der elementaren Mathematik. 2te Auflage. Leipzig, Dürr, 1914. 269 pp. M. 5.00

TOWNSEND (E. J.). Functions of a complex variable. (American mathematical series.) New York, Holt, 1915. 7+384 pp. $4.00

WILDBRETT (A.). Analytische Geometrie und Elemente der Differentialrechnung. Lehrbuch mit Aufgabensammlung für die Oberklasse von Gymnasien und Realgymnasien. Nürnberg, Korn, 1915. 124 pp. M. 2.00

ZEUTHEN (H. G.). See ENCYCLOPÉDIE.

ZORETTI (L.). See ENCYCLOPÉDIE.

II. ELEMENTARY MATHEMATICS.

BELL (H.). A course in the solution of spherical triangles for the mathematical laboratory. (Edinburgh Mathematical Tracts No. 5.) London, Bell, 1915. 8vo. 8+66 pp. 2s. 6d.

BEINHORN (H.). Lehrbuch der Mathematik. Berlin, Weidmann, 1915. Ausgabe A für Realanstalten, 4 Bände. 165+197+216+234 pp. M. 2.00+2.20+2.40+2.80. Ausgabe B für Gymnasialanstalten, 3 Bände. 274+175+211 pp. M. 3.00+2.20+2.40.

BORCHARDT (W. G.). Revision papers in algebra. London, Rivingtons, 1915. 2s.

BORCHARDT (W. G.) and PERROTT (A. D.). Key to geometry for schools. London, Bell, 1915. 294 pp. 8s. 6d.

CRANTZ (P.). Arithmetische Aufgaben für Oberlyzeen sowie die mittleren und oberen Klassen der Studienanstalten. 2te Auflage. Leipzig, Teubner, 1913. 88 pp. M. 1.40

DUPUIS (N. F.). Elementary synthetic geometry of the point line and circle, in the plane. 4th reprint. New York, Macmillan, 1914. 290 pp. $1.10

——. Elements of synthetic solid geometry. 3d reprint. New York, Macmillan, 1914. 12mo. 239 pp. $1.60

FISHER (G. E.) and SCHWATT (I.). Complete secondary algebra. 3d reprint. New York, Macmillan, 1914. 12mo. 504 pp. $1.35

GERBER (L.). Rechenunterricht und Krieg. Strassburg, Strassburger Druckerei und Verlagsanstalt, 1915. 80 pp. M. 1.00

GNAGA (A.). Sulla estrazione di radice quadrata e cubica col procedimento delle medie. Brescia, tip. F. Apollonio, 1914. 8vo. 15 pp.

HUN (J. G.) and MACINNES (C. R.). The elements of plane and spherical trigonometry. 3d reprint. New York, Macmillan, 1915. $0.90

INGOLD (L.). See KENYON (A. M.).

KENYON (A. M.) and INGOLD (L.). Plane and spherical trigonometry. 2d reprint. New York, Macmillan, 1914. 132 pp. $1.35

LESSER (O.). See SCHWAB (K.).

LISTER (S.). A first book of arithmetic. London, Macmillan, 1915. 28+239 pp. 1s. 6d.

MACINNES (C. R.). See HUN (J. G.).

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College, February–June, 1915. London, Macmillan, 1915. 8vo. 24 pp. 1s.

PERROTT (A. D.). See BORCHARDT (W. G.).

ROBBINS (E. R.). New plane geometry. New York, American Book Co., 1915. 6+264 pp.

ROTHROCK (D. A.). Elements of plane and spherical trigonometry. 7th reprint. New York, Macmillan, 1915. 147 pp. $1.10

SAMMLUNG mathematischer Formeln. Herausgegeben vom bayerischen Mathematiker-Verein. Ausgabe A für Gymnasien, Realgymnasien, Realschulen. Ausgabe B für Oberrealschulen. München, J. Lindauer, 1915.

SCHNEIDER (A.). See SCHWAB (K.).

SCHULTZE (A.). Advanced algebra. 12th reprint. New York, Macmillan, 1915. 12mo. 562 pp. $1.25

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk: A. Schneider, Lehr- und Uebungsbuch der Geometrie. Für Lehrer- und Lehrerinnenbildungsanstalten. 2ter Teil (Ebene Trigonometrie, Stereometrie und sphärische Trigonometrie). Leipzig, Freytag, 1915. 262 pp. M. 3.40

SCHWATT (I.). See FISHER (G. E.).

SMITH (C.). Elementary algebra. Complete edition, revised and adapted to American schools by I. Stringham. 5th reprint. New York, Macmillan, 1915. 12+672 pp. $1.20

STRINGHAM (I.). See SMITH (C.).

STUCKEY (J. J.). Table of compound interest at ½ per cent. and of antilogarithms to sixty figures to base 1.00125. London, G. Allen and Unwin, 1915. 4to. 116 pp. 1£ 1s.

TABLES for converting shillings and pence and farthings into 7 places of decimals of a pound and for the reconversion of decimals. London, Layton, 1915. 8 pp. 1s.

THAER (A.) und WIMMENAUER (T.). Arithmetische Aufgaben für höhere Schulen. Ausgabe B, für Oberrealschulen, Realgymnasien und verwandte Anstalten. 1ter Teil: Unterstufe; 2ter Teil: Oberstufe. Leipzig, Hirt, 1915. 146+152 pp. M. 1.75+1.75

URAGUCHI (Y.). Handy logarithmic tables. Tokyo, Y. Uraguchi, 1915. 12 sen

WENZEL (G.). Geometrie (Hilfsbücher zur Vorbereitung für die Bürgerschullehrerprüfung, V. Band). Wien, Tempsky, 1915. 404 pp. K. 5.00

WIMMENAUER (T.). See THAER (A.).

ZICHEROW (—.). Das abgekürzte Rechnen, für Eltern und Schüler dargestellt. Rawitsch, Birkenstock, 1915. 15 pp. M. 0.35

III. APPLIED MATHEMATICS.

ABETTI (A.). Osservazioni astronomiche fatte all'equatoriale di Arcetri nel 1914, ed appendice di M. Maggini. Firenze, tip. Galletti e Cocci, 1915. 4to. 105 pp.

ANDERSON (A. J.). The romance of Leonardo da Vinci. New York, Brentano, 1915. $3.75

ANNUARIO astronomico pel 1916, pubblicato dal r. osservatorio di Pino Torinese. Torino, tip. s. Giuseppe degli Artigianelli, 1915. 8vo. 165 pp. +1 tavola.

BARTON (E. H.). An introduction to the mechanics of fluids. London, Longmans, 1915. 14 +249 pp. 6s.

BIGOURDAIN (G.). Petit atlas céleste précédé d'une introduction sur les constellations, sur les moyens de les reconnaitre, . . . Paris, Gauthier-Villars, 1915. 60 pp. +5 cartes. Fr. 2.75

BLUMENTHAL (O.). See BORN (M.).

BORN (M.). Dynamik der Kristallgitter. (Fortschritte der mathematischen Wissenschaften in Monographien. Herausgegeben von O. Blumenthal. Heft 4.) Leipzig, Teubner, 1915. Gr. 8vo. 7 + 122 pp. M. 7.60

CASSINIS (G.). See REINA (V.).

CONNAISSANCE des temps ou des mouvements célestes pour le méridien de Paris à l'usage des astronomes et des navigateurs pour 1917 publiée par le Bureau des longitudes. Paris, Gauthier-Villars, 1915. 8vo. 30 +824 pp. Cartonné. Fr. 4.75

CONWAY (A. W.). Relativity. (Edinburgh Mathematical Tracts No. 3.) London, Bell, 1915. 8vo. 8 +43 pp. 2s.

COOK (T. A.). The curves of life, being an account of spiral formations and their application to growth in nature, to science and to art; with special reference to the manuscripts of Leonardo da Vinci. New York, Holt, 1915. Gr. 8vo. 30 +479 pp. $5.00

DARIÈS (G.). Cubature des terrasses et mouvement des terres. 2e édition. Paris, Gauthier-Villars, 1914. 8vo. 196 pp. Cartonné. Fr. 3.00

DREYER (I. L. E.). See TYCHONIS BRAHE.

GREENHILL (G.). Report on gyroscopic theory. (Advisory committee for aeronautics, Reports and memoranda No. 146.) London, Darling and Son, 1914. Folio. 277 pp. Boards. 10s.

GREGORY (R. A.) and HADLEY (H. E.). A manual of mechanics and heat. London, Macmillan, 1915. 8 +309 pp. 3s.

GROAT (B. F.). Chemi-hydrometry and its application to the precise testing of hydro-electric generators. (*Proceedings of the American Society of Civil Engineers*, vol. 41, No. 9, pp. 2103–2427.) New York, 1915.

HADLEY (H. E.). See GREGORY (R. A.).

HAUSSNER (R.). Darstellende Geometrie. 2ter Teil: Perspektive ebener Gebilde; Kegelschnitte. 2te verbesserte und vermehrte Auflage. Berlin, Göschen, 1914. M. 0.90

HIGBEE (F. G.). The essentials of descriptive geometry. New York, Wiley, 1915. 6+204 pp. $2.00

INCE (E. L.). A course of descriptive geometry and photogrammetry for the mathematical laboratory. (Edinburgh Mathematical Tracts No. 1.) London, Bell, 1915. 8vo. 8+79 pp. 2s. 6d.

JOHNSON (J. F.). Practical shop mechanics and mathematics. New York, Wiley, 1915. 9+130 pp. $1.25

KOPFF (M.). Die Bahnen der beweglichen Gestirne im Jahre 1915. Eine astronomische Tafel nebst Erklärung. Berlin, 1915. M. 0.40

LEONARDO DA VINCI. See COOK (T. A.)

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